For this chapter apart from video → try and cover all Q's as the mathematical formulations are very good. (including some vector illustrations) → section C *

Chapter → 10.

Linear programming.

It is a technique of decision making which is used in contradictory situations. Under this technique, following procedure shall be used.

Step 1: Determination of mathematical formulation.

Step 2: Solution of mathematical formulation by following two methods :-

(a) Graphical method – this method can be used when there are only two variables.

(b) Simplex method – It can be used when there are 2 or more variables.

Mathematical Formulation : Given problem can be converted into a mathematical format by following procedure :-

$3x_1 + 2y_1 \geq 16$, $x_1 + y_1 \geq 16$ → constraints

max: $6x_1 + 2y_1$ ← obj function.

$x_1, y_1 \geq 0$ → non negative constraints

Step 1:- Define variables ($x_1, x_2 \dots$)

Step 2:- Now, Define objective function (either maximization (if profit related) or minimization (if cost related))

Step 3:- Now, Define constraints for each limited resources & other conditions [$\geq$, $\leq$, $=$]

Step 4:- Now define non-negative constraints. $\in [x_1, x_2 \dots x_n \geq 0]$ (They should always be non negative)

(a) Graphical method :- This method can be used when there are only 2 variables (as many constraints). Under this method following procedure shall be used:-

(see page 204-205) of copy.

Step 1:- Determine mathematical formulation.

Step 2:- Now, convert each constraint (inequality) into an Equation.

Step 3:- Now, Determine value of each variables for each constraint by extreme Theorem. (considering "0")

Step 4:- Now, Draw/Plot each constraint on graph & determine feasible region.

Step 5:- Now, Determine value of each variable in feasible region to determine optimum solution.

(Points on boundary are more correct in terms objective, but point of intersection is best value / optimal / optimum solution.

see recorded video first on simplex only then do numericals for this part.

(b) Simplex method (maximisation case) :- Under this method following procedure shall be used.

Step 1:- Determine mathematical function.

Step 2:- Now, introduce slack variables ($s_1, s_2 \ldots s_n$) in each constraint & in objective function, multiple of slack variable in objective function shall be zero

(Value of slack variable is equal to diff at the end of solution)

Step 3:- Now construct initial simplex table as follows:-

Eg:-

maximize $= 3x_1 + 4x_2 + 0s_1 + 0s_2$

Subject to the constraints:-

$2x_1 + 3x_2 + s_1 = 16$

$4x_1 + 2x_2 + s_2 = 16$

$x_1, x_2 \geq 0$.

Initial simplex table.

Cs.	Solution variables	Solution quantity	x_1	x_2	s_1	s_2	Minimum Ratio.
	c_j		3	4	0	0	
0	s_1	16	2	3	1	0	16/3 → exi. v.
0	s_2	16	4	2 (a)	0	1 (b)	16/2
	z_j [$\Sigma C \times x_j$]		0	0	0	0	
	$c_j - z_j$ (opportunity cost)		3	4	0	0	

Key row

Key element

Key column

Introduce

0×2 + 0×4   0×3 + 0×2   0×1 + 0×0   0×0 + 0×1

After $c_j - z_j$:-

(i) now Identify key column :- column corresponding to the highest positive entry in $c_j - z_j$ row

(ii) now Determine minimum ratio as follows :-

$$\text{min Ratio} = \frac{\text{Solution Quantity}}{\text{corresponding element of key column}}.$$

(iii) now Identify key row :- the row corresponding to least positive minimum ratio

[positive zero = least positive
negative " = not considered.]

(iv) now Identify key element :- element at intersection of key row & column.

Step 4 :- now construct second simplex table as follows :-

Simplex Table - 2.

$c_j \rightarrow$			3	4	0	0	minimum
↓	sol'n v	sol'n Q.	x_1	x_2	S_1	S_2	ratio.
4	x	16/3	2/3	1	1/3	0	8
0	S_2	16/3	8/3	0	-2/3	1	2
	z_j		8/3	4	4/3	0	
	$c_j - z_j$		1/3	0	-4/3	0	

[b - a.x.e.]

Key element = 8/3.

(i) new element of key row shall be determined as follows :-

$$\text{new Element} = \frac{\text{Old Element}}{\text{Key Element}}$$

(ii) Other Elements shall be determined as follows:-

$$\text{New Element} = \text{Old Element} - \frac{\text{Corresponding Element of Key column} \times \text{Corresponding element of Key row}}{\text{Key element}}$$

If any element in Key row is zero, then element of corresponding column remain unchanged.

If any element in Key row is zero, then other element of corresponding row remain unchanged.

All elements of Key column becomes zero (except Key element)

Simplex Table – 3.

$c_j \rightarrow$			3	4	0	0	minimum
↓	Sol^n var	Sol^n Q.	x_1	x_2	S_1	S_2	ratio
4	x_2	4	0	1	1/2	-1/4	
3	x_1	2	1	0	-1/4	3/6	
	z_j		3	4	5/4	1/8	
	$c_j - z_j$		0	0	-5/4	-1/8	

So, no further steps, as $c_j - z_j$ is zero or negative.

Thus, 4 units of x_2 & 2 units of x_1

(b) (ii) Simplex Method (minimisation case) :- Under this method, following procedure shall be used.

Step 1: Determine Mathematical Formulation

Step 2: Now introduce surplus variables ($s_1, s_2 \ldots s_n$) & artificial variables ($A_1, A_2 \ldots A_n$) in each constraint & objective function.

In objective function multiple of artificial variable shall be infinite ("M") & multiple of surplus variable shall be zero.

Step 3: Now construct Initial simplex table as follows :-

Eg : minimize $= 3x_1 + 4x_2$

subject to the constraints :-

$$2x_1 + 3x_2 - S_1 + A_1 = 16$$
$$4x_1 + 2x_2 - S_2 + A_2 = 16$$
$$x_1, x_2 \geq 0.$$

minimize $= 3x_1 + 4x_2 + 0S_1 + 0S_2 + MA_1 + MA_2$

subject to the constraints :-

$$2x_1 + 3x_2 - S_1 + A_1 = 16.$$
$$4x_1 + 2x_2 - S_2 + A_2 = 16$$

Initial simplex Table.

$c_j \rightarrow$			3	4	0	0	M	M	minimum
↓	sol^n v	sol^n.	x_1	x_2	S_1	S_2	A_1	A_2	Ratio.
M	A_1	16	2	3	−1	0	1	0	8
M	A_2	16	4	2	0	−1	0	1	4.
z_j ($M\times2 + M\times4$)			6M	5M	−M	−M	M	M	
$c_j - z_j$			(3−6M)	(4−5M)	M	M	0	0	

After $c_j - z_j$:-

(i) Identify key column :- The column corresponding to highest negative value in $c_j - z_j$ row (highest in negative)

(ii) Remaining procedure is same (minimum ratio & key row & key element)

Step 4: Now construct simplex Table 2:-
(using same procedure as in maximisation)

Simplex Table-2.

$c_j \rightarrow$ ↓	Sol^n v	Sol^n Q	3 x_1	4 x_2	0 S_1	0 S_2	M A_1	M A_2	minimum ratio
M	A	8	0	2	-1	1/2	1	-1/2	4
3	x_1	4	1	1/2	0	-1/4	0	1/4	8
z_j			3	$2M+3/2$	$-M$	$\frac{M}{2}-\frac{3}{4}$	M	$-\frac{M}{2}+3/4$	
$c_j - z_j$			0	$4-2M-3/2$	M	$-M/2+3/2$	0	$M+M/2-3/4$.	

(new) Simplex Table 3.

$c_j \rightarrow$ ↓	Sol^n v	Sol^n Q	3 x_1	4 x_2	0 S_1	0 S_2	M A_1	M A_2	min Ele.
4	x_2	4	0	1	-1/2	1/4	1/2	1/4	
3	x_1	2	1	0	1/4	-1/8	-1/4	3/8	
z_j			3	4	-5/4	-3/8	5/4	1/8	
$c_j - z_j$			0	0	5/4	3/8	$M-5/4$	$M-1/8$.	

So, no further steps as $c_j - z_j$ is zero or positive.
Thus 4 units of x_2 & 2 units of x_1

[Minimisation case] → only what is in " "

mixed Type constraints [maximisation case] :-

1. "If constraint is "less than" or "equal to" type, then only slack variable ($s_1, s_2 \ldots s_n$) shall be introduced."

2. "If constraint is "equal to" type, then only Artificial variable shall be introduced" but multiple of Artificial variable in objective function shall be taken "$-M$"

3. "If constraint is "equal to" or "greater than" type, then surplus & Artificial variable shall be introduced" but multiple of Artificial variable in objective function shall be "$-M$"

in objective function, multiple of Ar (shall be "$-M$")

Dual of Linear programming :- when maximisation problem is converted into minimisation problem or vice-versa.

when primal is converted into Dual, following conditions must be satisfied :-

(i) if obj function is in max" → all constraints must be $\leq$ type.

(ii) if one or more constraint is $\geq$ type, then it must be converted into "$\leq$" type.

Ex:- $5x_2 + 4x_2 \geq 100 \rightarrow -4x_1 - 5x_2 \leq 100$

(iii) If obj function is mini", then → all constraints must be $\geq$ type. If one or more constraint is $\leq$ type then it must be converted into $\geq$ type

(iv) If any constraint is "=" type, then it shall be replaced by two constraints, one is $\leq$ and other is $\geq$

Ex:- $4x_1 + 3x_2 - x_3 = 4$.

$4x_1 + 3x_2 - x_3 \geq 4$.

$4x_1 + 3x_2 - x_3 \leq 4$

(v) If any variable is unrestricted in sign then it shall be replaced by diff of two positive variables (integers)

Primal shall be converted into Dual as follows:-

Primal		Dual
1. maximisation	$\longrightarrow$	minimisation
minimisation	$\longrightarrow$	maximisation
2. no. of. variables	$\longrightarrow$	no. of. constraints
3. no. of. constraints	$\longrightarrow$	no. of. variables
4. $\leq$	$\longrightarrow$	$\geq$
$\geq$	$\longrightarrow$	$\leq$
5. $x_1, x_2 \dots x_n$	$\longrightarrow$	$y_1, y_2 \dots y_n$

Ex:-

primal: Maximise $Z = 13x_1 + 12x_2$

Subject to $x_1 - 4x_2 \leq 10$

$2x_1 + 3x_2 \leq 5$;

$-2x_1 - 3x_2 \geq -5$

Dual: Minimise $Z = 10y_1 - 5y_2$

Subject to $(y_1 - 2y_2) \geq 13$

$(-4y_1 - 3y_2) \geq 12$

Solution types :-

1. Feasible solution $\rightarrow$ no artificial variable in Basic variable column.
2. Optimal solution $\rightarrow$ $C_j - Z_j \leq 0$ in maximisation problem or vice-versa (in minim^n)
3. multiple optimal solution $\rightarrow$ non Basic variable have "0" in $C_j - Z_j$
4. Degenerate solution $\rightarrow$ If any basic variable have "0" in Quantity