

CHAPTER 11: LINEAR PROGRAMMING

Question:

Explain LPP.

Answer:

Linear programming is a mathematical technique for determining the optimal allocation of resources and achieving the specified objective when there are alternative uses of the resources like money, manpower, materials, machines and other facilities. The objective in resource allocation may be either cost minimization or profit maximization.

Categories of the Linear Programming Problems (LPP):

- i. General Linear Programming Problems.
- ii. Transportation Problems.
- iii. Assignment Problems.

(General Linear Programming problems are dealt with in this chapter and the rest will be taken up in the following chapters.)

General Linear Programming: A linear programming problem consists of an objective function (viz. Maximising or Minimising) with a set of variables subject to certain constraints involving the usage of resources that can be expressed as linear mathematical functions.

Question:

Explain the requirements of LP.

Answer:

In order to apply LP the following requirements are to be met:

- a. **Objective to be identifiable and measurable:** There should be an objective which should be in identifiable and measurable terms.
- b. **Activities to be identifiable and measurable:** The activities to be included should be distinctively identifiable and measurable in quantitative terms.
- c. **Resources to be identifiable and measurable:** The limited resources of the system which are to be allocated for attainment of goal should also be identifiable and measurable.

- d. **Divisibility:** The resources required are directly proportional to respective outputs.
- e. **Additivity:** The relationships representing the objective function and the resource constraints must be in linear nature in the form of equations or inequalities respectively.
- f. **Finite Choices:** There should be feasible alternative courses of action available to the decision maker.

When the above conditions are satisfied in a given situation, the problem can be expressed in algebraic form called LPP and then solved for optimal solution.

Formulation of an LPP: The basic and first step for a solution to a LPP begins with the formulation of the objective function with variables and utilisation of the resources with respective constraints expressed in the form of equations or inequations. This can be explained further by an example.

Consider a unit manufacturing 2 products A & B with unit profits of Rs. 3 & Rs. 4 respectively. Further, the products are to be produced using 2 materials X & Y as per the following table:

Particulars	Material X	Material Y
Product A	2 Kgs.	4 Kgs.
Product B	3 Kgs	2 Kgs.
Maximum availability of materials	16 Kgs.	16 Kgs.

Presuming that there is no restriction on demand, and x_1 & x_2 are the respective units of A & B produced, the above can be formulated as:

Maximise $3x_1 + 4x_2$; Objective function

Subject to:

$$2x_1 + 3x_2 \leq 16 \quad \text{Constraint for Material X}$$

$$4x_1 + 2x_2 \leq 16 \quad \text{Constraint for material Y}$$

Non Negativity Constraint: Further, the units of A & B produced cannot be negative. Hence the following non negativity constraints are also included.

$$x_1 \geq 0 \text{ \& } x_2 \geq 0$$

Solution of an LPP: There are 3 methods of solving an LPP:

- a. Graphical Method;
- b. Trial and Error Method; and

c. Simplex Method.

Graphical Method: This method can be used to solve LPP only when there are 2 variables. For higher variables, this method cannot be applied. The following are the steps involved in solving LPP by graphical method:

- a. Formulating the linear programming problem.
- b. Plotting the capacity constraints on the graph paper.
- c. Determining the region that satisfies the set of given inequalities.
- d. Ensuring that the feasible region is bounded. If the region is not bounded, it implies either there are additional hidden conditions or the problem does not have solution.
- e. Identifying feasible region and coordinates of corner points.
- f. Constructing the matrix E of the extreme points, and the column vector C of the objective function.
- g. Testing the corner point which gives maximum profit. The optimum solution to a LPP will lie only at one of the corner points only based on **EXTREME POINT THEOREM**. Hence, other intermittent points need not be checked.
- h. Finding the matrix product E C. The objective function is optimised relating to the same row elements of the extreme point matrix E.
- i. If the slope of the objective function is same as that of one side of feasible region, there are multiple solutions to the problem. However, the optimized value of the objective function remains the same.
- j. For decision – making purpose, sometimes, it is required to know whether optimal point leaves some resources unutilized.

Consider:

Maximise $3x_1 + 4x_2$; Objective function

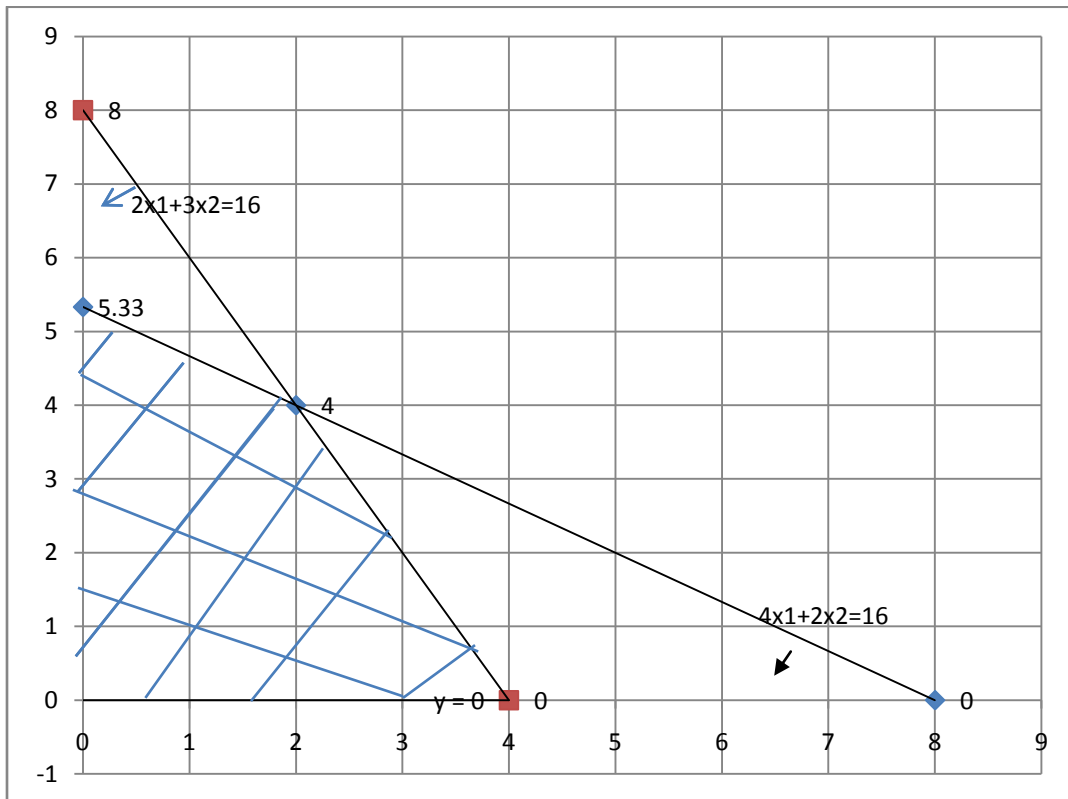
Subject to:

$2x_1 + 3x_2 \leq 16$ Constraint for Material X

$4x_1 + 2x_2 \leq 16$ Constraint for material Y

$x_1, x_2 \geq 0$

The graph will be:



The shaded region in the graph represents the feasible region. Hence the solution will be at one of the corners of the feasible region multiplied with column vector Viz.

$$\begin{pmatrix} 0 & 0 \\ 0 & 5.33 \\ 2 & 4 \\ 4 & 0 \end{pmatrix} \times \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

Thus the values will be:

For (0, 0) it will be $0*3+0*4=0$

For (0, 5.33) it will be $0*3+5.33*4=21.32$

For (2, 4) it will be $2*3+4*4=22$ and

For (4, 0) it will be $4*3+0*4=12$

Of the above, since 22 is maximum, it is the result.

Trial & Error Method: This is the algebraic approach of solving LPP. Under this method, first the inequalities are to be converted into equalities. This can be done by adding non negative slack variables in the equations. **Slack variables represent idle resources.** In the objective function, the contribution per unit of a slack variable is always taken as zero, since no profit can be made on idle resources. Upon adding slack variables, the LPP formulation illustrated above will be:

Maximise $3x_1 + 4x_2 + 0x_3 + 0x_4$; Objective function

Subject to:

$$\left. \begin{array}{l} 2x_1 + 3x_2 + x_3 + 0x_4 = 16 \text{ Constraint for Material X} \\ 4x_1 + 2x_2 + 0x_3 + x_4 = 16 \text{ Constraint for material Y} \end{array} \right\} \mathbf{A}$$

$$x_1 \geq 0; x_2 \geq 0; x_3 \geq 0; x_4 \geq 0.$$

Similarly in $\geq$ (greater than equal to) type inequalities, we subtract a variable called surplus variable to convert into equality. **Surplus variables represent excess amount of resources utilised over and above the available resources.** In the objective function, the contribution per unit of surplus variable is also taken as zero.

In the equations mentioned above, the number of variables are greater than the number of equations. These types of equations give infinite solutions yet it has finite vertices. The coordinates of the vertices can be determined by applying Basis Theorem.

Basis Theorem: It states that for a system of m equations in n variables (where $n > m$) has a solution in which at least $(n-m)$ of the variables have value of zero as a vertex. This solution is called a basic solution.

In our illustration, we are having 4 variables with 2 equations. Hence, as per basis theorem, out of 4 variables, at least 2 variables should have zero values. By permutation and combination method, assigning zeros to 2 variables at a time in the given set of 2 equations of the illustration, we get below the 6 sets of simultaneous equations:

Set 1	When x_1 & x_2 are taken as zeros:	$1x_3 + 0x_4 = 16$	$\rightarrow$	$x_3 = 16$
		$0x_3 + 1x_4 = 16$	$\rightarrow$	$x_4 = 16$
Set 2	When x_1 & x_3 are taken as zeros:	$3x_2 = 16$	$\rightarrow$	$x_2 = 16/3$
		$2x_2 + x_4 = 16$	$\rightarrow$	$x_4 = 16/3$
Set 3	When x_1 & x_4 are taken as zeros:	$3x_2 + x_3 = 16$	$\rightarrow$	$x_3 = -8$

		$2x_2=16$		$x_2=8$
Set 4	When x_2 & x_3 are taken as zeros:	$2x_1=16$	➡	$x_1=8$
		$4x_1+x_4=16$	➡	$x_4=-16$
Set 5	When x_2 & x_4 are taken as zeros:	$2x_1+x_3=16$	➡	$x_3=8$
		$4x_1=16$	➡	$x_1=4$
Set 6	When x_3 & x_4 are taken as zeros:	$2x_1+3x_2=16$	➡	$x_1=2$
		$4x_1+2x_2=16$	➡	$x_2=4$

The equations are solved as simultaneous equations to get the values of variables. Since set 3 & set 4 have negative values (which is against our assumption of ≥ 0) they are ignored.

By substituting the values of x_1 to x_4 in the objective function:

Maximise $3x_1+4x_2+0x_3+0x_4$, we get

For set 1	0
For set 2	64/3
For set 5	12
For set 6	22

Hence, solution for set 6 is optimal.

Limitations of Trial & Error Method: This method has serious limitations as detailed below:

- In case the constraints (m) and variables (n) are more, the solution will be very tedious and time consuming.
- The profits / losses of successive solutions keep fluctuating as seen above.
- Since some sets yield unfeasible solutions, there needs to be a method for their early identification and elimination to save time.

Simplex Method: This is a mathematical algorithm for solving LPP and is very widely used. In this case, subsequent iterations lead to successive improvements in arriving at the objective of maximisation or minimisation. This is highly efficient and versatile and also amenable for further mathematical treatment and interesting interpretations can be made. The simplex algorithm applies to both maximisation and minimisation problems. The only difference in the algorithm involves the selection of the incoming variable. In the maximisation problem the incoming variable is the one with highest +ve value in net evaluation row (NER). (Conversely, it is the most -ve variable that is selected as the incoming variable in a minimization problem.) And if all elements in the NER are either negative (or +ve for

minimisation) or zero, it is the indication for the optimal solution.
Considering the initial example:

Maximise $3x_1 + 4x_2 + 0x_3 + 0x_4$; Objective function

Subject to:

$$\left. \begin{array}{l} 2x_1 + 3x_2 + x_3 + 0x_4 = 16 \text{ Constraint for Material X} \\ 4x_1 + 2x_2 + 0x_3 + x_4 = 16 \text{ Constraint for material Y} \end{array} \right\} \mathbf{A}$$

$x_1 \geq 0; x_2 \geq 0; x_3 \geq 0; x_4 \geq 0.$

The initial Simplex table can be formed as below:

Coefft. Matrix					Identity Matrix				
Basic Variables		Non Basic Variables				Identity Matrix			
	1	2	3	4	5	6			
	Fixed Ratio	Program (basic variables)	Profit / unit	Qty.	3	4	0	0	Replac ement Ratio
		x3			x1	x2	x3	x4	
A		x3	0	16	2	3	1	0	16/3
	2/3	x4	0	16	4	2	0	1	8
		NER			3	4	0	0	
B	1/4	x2	4	16/3	2/3	1	1/3	0	8
		x4	0	16/3	8/3	0	-2/3	1	2
		NER			1/3	0	-4/3	0	
C		x2	4	4	0	1	1/2	-1/4	
		x1	3	2	1	0	-1/4	3/8	
		NER			0	0	-5/4	-1/8	

Simplex table is vertically divided into 6 columns (1,2,3,4,5&6) and horizontally into 3 rows A, B & C.

Column 1 consists fixed ratio that is obtained by dividing the corresponding key column element with key element.

Column 2 consists basic variables that are considered for the solution (Basic variables are the variables that are listed under program column. Rest of the variables are called non-basic variables.) In the initial solution, always, artificial slack variables if any, and surplus and / or slack variables are considered.

Column 3 consists corresponding coefficients of the basic variables in the objective function.

Column 4 consists figures listed on the right hand side of the constraints.

Column 5 consists respective coefficients of the objective function.

Column 6 consists Replacement Ratio that is obtained by dividing the quantities with respective elements in the key column.

Figures in rows A, B & C indicate successive iterations.

The steps involved in Simplex Method solution are:

- a. **Formulation of LPP** by restating in mathematical form, i. e writing the objective function and the constraints;
- b. **Developing equations** from inequalities by adding or deducting slack / surplus / artificial slack variables;
- c. **Ensuring all variables are ≥ 0** : All variables are to be ≥ 0 . If there is any unrestricted variable (discussed later), it should be converted.
- d. **RHS of the constraints to be +ve**: It should be ensured that the right hand side of the constraints is +ve. If not, it should be made +ve by multiplying the entire equation with -1.
- e. **Developing the initial table** including the NER;
- f. **Identifying the Key Column**: Key column is the column with highest +ve no. for maximisation problems and highest -ve no. for minimisation problems from the values in NER;
- g. **Identifying the Key Row**: Key row is the minimum of the figures obtained by dividing the quantities with respective elements of pivot column. The figures so obtained are called Replacement Ratios or Minimum Ratios. In case if there happen to be any negative figures as replacement ratios, such figures are to be ignored. However, zeros are to be considered. Key Element (i.e. the intersection element of Key column and Key Row);

- h. **Calculating the revised row:** This is calculated by dividing all the elements of key row with the Key Element.
- i. **Calculating the Fixed Ratio:** This is arrived by dividing each element of Key Column with Key Element;
- j. **Calculation of balance rows:** This is done by subtracting the existing row element from the product of the fixed ratio and the corresponding key element in the key row;
- k. **Ascertaining the NER:** This is done by deducting the sum of the products of profit column figures with corresponding elements in each column and deducting such sum from the corresponding coefficient of the objective function;
- l. **Checking for optimality:** i. e. to find whether all the values in the NER are zero or are -ve for maximisation objective (zero or +ve for minimisation objective.)
- m. If this is not achieved then, steps (d) to (j) are to be repeated till these criteria are satisfied.

The words Key and Pivot are used interchangeably with same meaning.

Note: We understand that the mathematical language and sense of the points described above are confusing and could be difficult to understand as well at one outgo. Readers are strongly advised to very carefully follow the method of solving the LPP illustrations in the class room and to practice them to understand and appreciate the beauty of this versatile mathematical algorithm backed by strong logic.

Artificial Slack variables: consider the constraint function $3x+9y \geq 100$. To convert this into equal, we deduct from left hand side the Surplus Variable (S1) thus making it $3x+9y-S1=100$. **Surplus variables represent excess amount of resources utilised over and above the available resources.** In case we presume x & y to be zero, then the value of S1 turns to be -ve. This will be against our basic assumption of all variables ≥ 0 . To overcome this contradiction, we introduce another variable called Artificial Slack Variable A1. **Artificial Variables represent imaginary brands.** Whereas slack variables and surplus variables have always zeros as cost coefficients, Artificial slack variables always have infinitely large cost coefficients, usually represented by M. Further, the sign of artificial slack variables in the objective function depends on the type of objective function whether it is maximising or minimising. In case of maximising problems, the sign of M will be -ve and for minimising problems, the sign will be +ve. The signs for

artificial slack variables in objective function do not have any relationship with the signs $\geq$ or $\leq$ in the constraint function. Further, in the initial iteration, always artificial slack variables are considered in the program column and once artificial variables are replaced with real variables, they will never come into the iteration again due to the infinitely large cost coefficients associated with them.

Equality sign in constraints: In case there exists '=' (equal) sign in the constraint, then only Artificial Slack Variable is added with M as coefficient in the objective function. Sign of M in objective function depends on the type of problem (i.e. maximisation (-ve) or minimisation (+ve)). Slack or surplus variables are not used for this constraint. This is because under simplex method, in the initial solution, only slack variables, surplus variables and artificial slack variables are considered for iteration.

Question:

Define and explain unrestricted variable.

Answer:

Unrestricted variable: One of the primary conditions of an LPP is all variables should be $\geq$ zero. There could be cases where a variable can take any value viz. -ve, zero or +ve. This type of variable is called unrestricted variable. In such cases, the unrestricted variable is to be shown as the difference of 2 non -ve variables, thus meeting the requirement of LPP.

Example: consider the objective function having 3 variables $x_1, x_2 \geq 0$ and x_3 is unrestricted. x_3 is unrestricted implies it can be -ve, zero or +ve. In these types of cases, x_3 should be represented as $x_4 - x_5$, where both $x_4, x_5 \geq 0$ and x_3 should be replaced by $x_4 - x_5$ in the objective function and all constraints. After arriving at the solution, at the end, $x_4 - x_5$ should be substituted for x_3 .

Question:

Explain Multiple Optimal Solutions with graphical illustration:

Answer:

The solution to a LPP need not be unique. An LPP may have multiple optimal solutions, and this will happen when:

1. One of the constraints line is parallel to objective function line on graph (i.e. iso profit line, a line indicating same profit.),
2. the constraint line should form part of boundaries of feasible region of the graph, and
3. the constraint should be a binding constraint (Binding constraint is a constraint that is essential for the given problem. ie. It cannot be redundant constraint.)

In the optimal simplex table, if the NER contains zero(s) under non basic variable(s), then the solution is not unique and multiple solutions do exist.

Consider maximise $8x_1 + 16x_2$

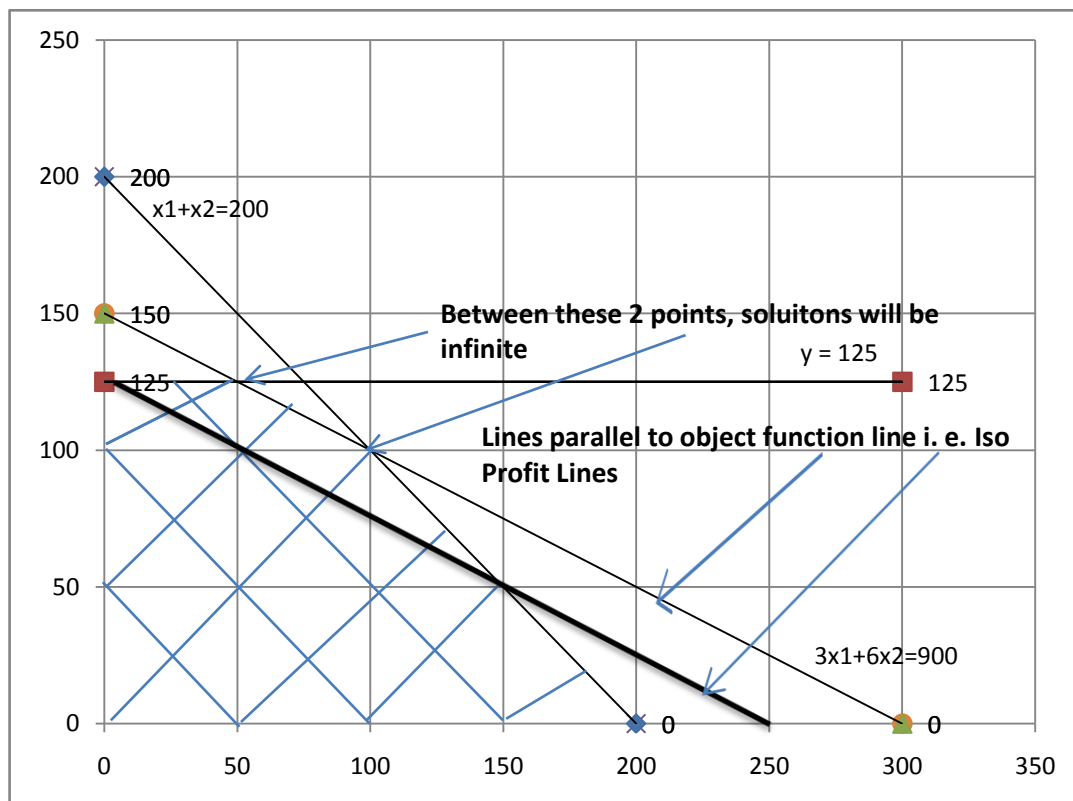
Subject to $x_1 + x_2 \leq 200$

$x_2 \leq 125$

$3x_1 + 6x_2 \leq 900$ and

$x_1, x_2 \geq 0$

The graph for the above will be as below:



It may be observed that the points (50, 125) and (100, 100) give same results for the objective function. Further, infinite solutions will exist between these two points.

Question:

Explain Infeasible Solution with graphical illustration.

Answer:

There will not be any solution to an LPP when the constraints are inconsistent. In the graphic method we can find this when the feasible region is empty and unbounded. i.e. there will not be any point on the graph which meets all constraints.

When there exists an Artificial variable as basic variable with a +ve value in quantity column of optimal simplex table, then there will be no feasible solution.

Consider maximise $20x_1 + 30x_2$

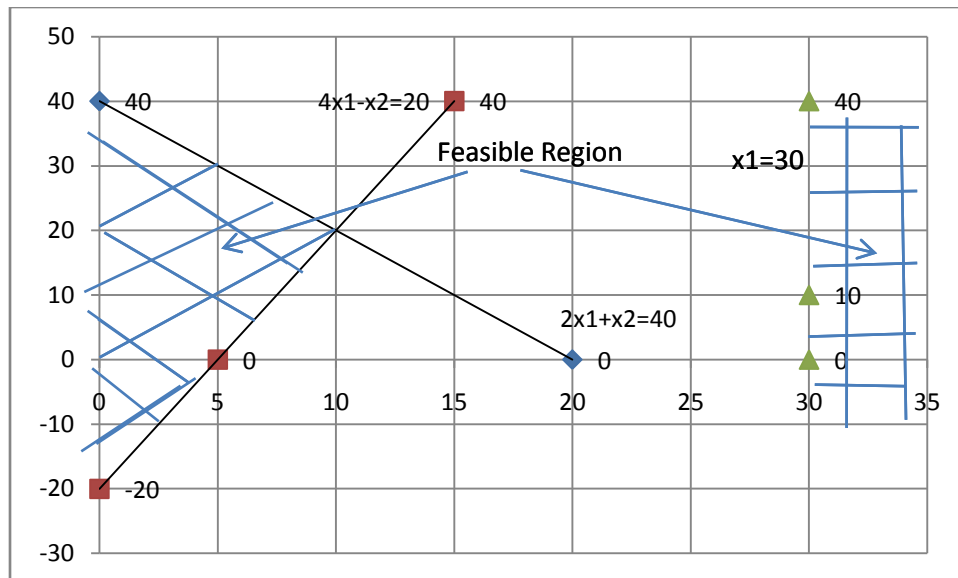
Subject to $2x_1 + x_2 \leq 40$

$4x_1 - x_2 \leq 20$

$x_1 \geq 30$ and

$x_1, x_2 \geq 0$

This can be graphically represented as:



Since there do not exist any points in the graph satisfying all the given constraints, the solution is infeasible.

Question:

Explain Unbounded Problem with graphical illustration.

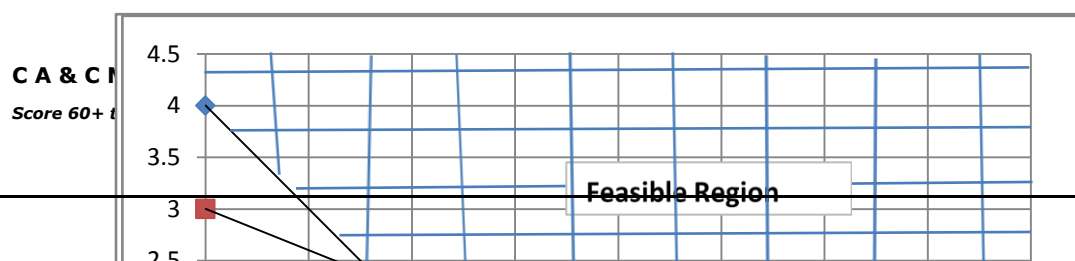
Answer:

For a maximisation LPP unboundedness occurs when there is no constraint in the solution so that one or more variables can be increased infinitely without violating any of the constraints. It could be possible to find several high values to variables obeying the constraints.

If all replacement ratios are $-ve$ or equal to ∞ in a simplex table, then the algorithm terminates and it implies the solution is unbounded.

Consider maximise $10x_1 + 20x_2$
 Subject to $2x_1 + 4x_2 \geq 16$
 $x_1 + 5x_2 \geq 15$ and
 $x_1, x_2 \geq 0$

this can be graphically represented as:



In this case there is no outer limit to the feasible region. Hence, the solutions are infinite.

Infeasibility Vs. Unboundedness: Both infeasibility and unboundedness are similar as both do not have any specific optimal solution. The striking difference is in case of infeasibility there will not be even a single feasible solution whereas in case unboundedness there will be infinite feasible solutions.

Degeneracy: When one or more of the basic variables have zero in quantity column, the simplex table and the solution are said to degenerate. This happens when in the preceding table the replacement ratios of 2 or more basic variables are same. Further, in case of degeneracy, the following table will not reflect any improvement in the objective function, which is one of the main features of simplex tables. In such a scenario, the table will not comply with Basis Theorem. It is very important to note that degeneracy in LPP could be temporary and could vanish in the final solution. Hence the table should be solved till NER criteria are met.

In graphic method it can be identified when one of the constraint lines does not pass through the optimum coordinates.

Question:

Write the characteristics of Dual in Linear Programming.

Answer:

1. Every LPP is called Primal and can be expressed as a dual and vice versa.
2. The number of constraints in the primal model equals the number of variables in the dual model and vice versa. Further, the coefficients of the objective function in the primal become the right hand sides of the constraints in the dual and the right hand sides of the primal become the coefficients of the objective function in the dual. Vice versa also holds good between dual and primal.
3. If the primal model is a maximisation problem then the dual will be a minimisation problem and vice versa.
4. If the constraints in primal have $\leq$ sign, in the dual they have $\geq$ sign and vice versa. Before writing dual it is necessary to express the primal LPP in standard form. i.e. all the constraints for a maximisation problem are to be put in the form of $\leq$ and for minimisation problem all the constraints are to be put in the form of $\geq$. All variables for the problem should be non -ve. i.e. $\geq$ zero.

A $\geq$ sign can be converted into $\leq$ by multiplying both sides with -1 and vice versa.

Further, in case there is a **constraint with equality sign** it needs to be split into $\leq$ and $\geq$ signs constraints and multiplying one of them with -1 as per requirement depending on maximisation or minimisation.

Example:

Consider $3x_1 + 4x_2 = 22$.

This can be written as $3x_1 + 4x_2 \leq 22$ and $3x_1 + 4x_2 \geq 22$ and converting one of them into $\leq$ or $\geq$ sign by multiplying with -1 on either sides of the inequality depending on the requirement of maximisation or minimisation.

5. The solution of the primal model will be same as the solution of the dual model and vice versa.
6. The objective functions of the two optimal tables will have identical final values.
7. Dual of the primal's dual problem is the primal problem itself.

8. Feasible solutions to a primal and dual problem are both optimal if the complementary slackness conditions hold, that is, (value of a primal variable) $\times$ (value of the corresponding dual surplus variable) = 0 or (value of a primal slack variable) $\times$ (value of the corresponding dual variable) = 0. If this relationship does not hold, then either the primal solution or the dual solution or both are not optimal.
9. If the primal problem has no optimal solution because of infeasibility, then the dual problem will have no optimal solution because of unboundedness and vice versa.

Special points on LPPs':

1. The positive figures (for maximisation problems) in NER indicate the unit opportunity cost being foregone by not including them respectively in the program and vice versa for minimisation problems.
2. Surplus variable along with Artificial Slack variable are both used when we come across $\geq$ sign in the constraints functions to make them equal. Further artificial slack variable is used to comply with non negative assumption.
3. Fractions in Simplex iterations are to be continued as they are for ease in further workings instead of converting them into decimals. Converting them into decimals will land the solver into confusion and problems.
4. **Inequalities in wrong direction:** Whether to introduce slack or surplus and artificial slack variable depends on the type of inequality and has got nothing to do with whether the objective function is maximisation or minimisation.
5. **Sign of Artificial slack variable:** Similarly, The sign for artificial slack variables in objective function does not have any relationship with the signs $\geq$ or $\leq$ in the constraint function. In maximisation problems M has -ve sign and minimisation problems M has +ve sign. Once an artificial slack variable exits from simplex iteration, again it will never enter because of the prohibitively high value associated with it.
6. In case 2 or more variables have same values in NER then, any one of them can be chosen as incoming variable for iteration.
7. Lower or upper bounds can be specified in an LPP. For example, it can be given that variable $x_1 \geq 50$. In such cases, another variable y_1 is

assumed where $y_1 = x_1 + 50$ implies $x_1 = y_1 - 50$ and substituting the value of x_1 with $y_1 - 50$ at all places and continuing in the routine way.

8. In all simplex tables there is bound to be a unit matrix eventhough, the columns may not be adjacent.
9. Simplex method, Dual method, Graphical method and Trial & error methods provide different ways of solving the problems. In all cases the result will be same and each has its ads and disads. Of all, the simplex method is versatile.

Question:

Explain the areas where LP is used.

Answer:

LP can be comfortably used in:

- a) Production, Planning and Product Mix Problems;
- b) Blending Problems;
- c) Diet Problems;
- d) Trim-loss Problems;
- e) Distribution Problems;
- f) Advertising Mix;
- g) Manufacturing Problems;
- h) Assembling Problems;
- i) Investment Problems;
- j) Agricultural Applications;
- k) Flights scheduling;
- l) Production Balancing, Inventories, Work force,
- m) Personnel Assignment Problems; etc.

Question:

State the limitations of LPP.

Answer:

The limitations of LPP are:

1. Primary requirement of LPP is that objective function and the constraints are to be linear.

2. In LPP fractional values are permitted to the decision variables. Practically this may not be always possible. In certain cases this can be overcome by treating the fractional parts as Work in Process or rounding off fractions.
3. In LPP, coefficients of the objective function and the constraint equations are to be completely known and these should not change during period of study.
4. LPP does not consider the effect of time and uncertainty.
5. Parameters appearing in the LPP are presumed to be constant. Practically this may not be so.
6. LPP deals with single objective. For multiple objectives, Goal Programming and objective programming tools are used.