2 Pricing of Financial Instruments

These lecture notes will introduce and explain the use of financial instruments involved in investing, lending, and borrowing money. We will focus particularly on bonds (government and corporate), interest, future and present value of financial instruments, annuities and mortgage, and a few aspects of stocks (change in value, **dividends**, and **short selling**).

Direct Financial Instrument: Contract between initial saver and borrower. The two main types of direct financial instruments are **stocks** and bonds.

2.1 Bond Pricing

Savers: Spend less than their current income (A saver purchases a bond).

Borrowers: Spend more than their current income (Issue bond).

In the case of a **bond**, a **saver** is willing to purchase a bond from a **borrower**. After the exchange of a bond, the borrower has more cash now to make whatever purchase he deems necessary. The saver is not planning on spending all of his money anyway and purchases a bond with his excess cash. This guarantees the saver will receive that cash, plus interest, at a set date in the future.

A saver could be an individual, firm, or government whose income exceeds its expenditures. A borrower could be a firm looking to build new factories or a government looking to finance a war or new civil projects among various other examples.

Bond: A **security** that gives the owner the right to a fixed payment, at a predetermined future date

Maturity: The future date at which a bond is repaid

Principal: Payment promised on a bond at its maturity

A bond is sold by the **debtor** for a price P_0 at time $t = 0$. The **creditor** pays for the bond at time $t = 0$ under the agreement that he will be repaid at the price P_T at time $t = T$. This price and time are predetermined and are not subject to change.

Present Value: Price at which financial instrument is traded at time $t = 0$

Future Value: Price at which financial instrument is traded at time $t > 0$. For bonds, this time $t > 0$ is set and the future value is agreed upon at $t = 0$. For other financial instruments, this is not necessarily the case.

Liquid Bond: A bond that trades between $t = 0$ and established $t = T$. (They have low transaction fees.) There are several types of bonds classified by the borrowers of funds and the terms of maturity:

1. **U.S. Government Bonds**: Issued when government expenditures are greater than tax revenue

   Bills: $T < 1$ year to maturity. These bonds have no coupon payments. They are called discount bonds because $P_0 < P_T$

   Notes: $1 \leq T \leq 10$ years. Include coupon payments every 6 months. Last payment includes coupon payment and principal.

Bonds: $T > 10$ years. Include coupon payments every 6 months. Last payment includes coupon payment and principal.

Coupon Payment: Set at $t = 0$ based on current ($t = 0$) interest rate.

2. **Corporate Bonds**: Issued by corporations to pay for large investments and large short term payments.

   Commercial Paper: $T < 1$ year. These are short term bonds sold for immediate investments by companies, and they are sold at a discount from the principal price listed.

   Bond: $T \geq 1$ year. Issued by a corporation

Corporate bonds have higher interest rates and generally shorter terms for maturity than government bonds because they are riskier investments. Obviously, smaller firms generally are more likely to go **bankrupt** than larger, more established firms, but there is always a chance that any firm, even seemingly more stable ones, could unexpectedly go bankrupt.

A higher likelihood of bankruptcy means a firm's bonds are sold with a higher interest rate. Because the government is much less likely to default on its bond payments, government bonds are considered less risky and have lower interest rates and longer maturity terms.

2.2 Interest Rates

In this section, we will discuss the details of interest rates on investments. We use the notation:

- PV = Present Value =Initial value of investment=Principal(=P)
- FV = Future Value =Terminal value of investment(=A).
- r= **Annual interest**=Annual return made on investment, which is defined by:

$$\boxed{r \doteq \frac{FV - PV}{PV}} \tag{2.1}$$

Therefore, the terminal value FV of an initial investment PV after 1 year is

$$FV = PV + rPV,$$

or

$$\boxed{FV = PV(1 + r).} \tag{2.2}$$

When PV= \$1 then:

$$\begin{aligned} FV &= PV(1 + r) \\ &= 1 + r \\ &= PV + r \\ &= PV + \text{Effective Annual Rate} \end{aligned}$$

Annual Compounding for t years: If an initial amount PV is invested for t years at an interest rate r compounded annually, then applying formula (2.2) repeatedly we get

$$\boxed{FV = PV(1 + r)^t.} \tag{2.3}$$

Semiannual Compounding for t years at annual rate r: Semiannual compounding means that every 6 months investment PV earns interest $PV \cdot (r/2)$, which is reinvested. Therefore, after 6 months we have $FV = PV\left(1+\frac{r}{2}\right)$ and after 1 year we have the terminal value

$$\begin{aligned} FV &= PV\left(1+\frac{r}{2}\right)\cdot\left(1+\frac{r}{2}\right) \\ &= PV\left(1+\frac{r}{2}\right)^2, \end{aligned}$$

which at the end of 2 years becomes

$$\begin{aligned} FV &= PV\left(1+\frac{r}{2}\right)^2\cdot\left(1+\frac{r}{2}\right)^2 \\ &= PV\left(1+\frac{r}{2}\right)^{2\cdot 2}. \end{aligned}$$

Similarly working, we find that at the end of 3 years this investment becomes

$$FV = PV\left(1+\frac{r}{2}\right)^{2\cdot 3}.$$

Finally, after t years of semiannual compounding we obtain the terminal value:

$$\boxed{FV = PV\left(1+\frac{r}{2}\right)^{2t}} \tag{2.4}$$

Quarterly compounding for t years at annual rate r: Quarterly compounding means that every 3 months an investment PV earns interest $PV \cdot (r/4)$, which is reinvested. Following a similar to the case of semiannual compounding argument, we find that after t years of quarterly compounding we obtain the terminal value:

$$\boxed{FV = PV\left(1+\frac{r}{4}\right)^{4t}} \tag{2.5}$$

Furthermore, we obtain the following formlas for monthly, weekly and daily compounding.

Monthly Compounding for t years at annual rate r:

$$\boxed{FV = PV\left(1+\frac{r}{12}\right)^{12t}} \tag{2.6}$$

Weekly Compounding for t years at annual rate r:

$$\boxed{FV = PV\left(1+\frac{r}{52}\right)^{52t}} \tag{2.7}$$

Daily Compounding for t years at annual rate r:

$$\boxed{FV = PV\left(1+\frac{r}{365}\right)^{365t}} \tag{2.8}$$

Formulas (2.3) –(2.8) above are special cases of the following general case when the **compounding frequency** m takes the values: $m = 1, 2, 4, 12, 52, 365$.

Compounding with Frequency m per year for t years at annual rate r: This means that every $1/m$ portion of the year an investment PV earns interest $PV \cdot (r/m)$, which is reinvested. Therefore, after t years of such compounding an initial investment PV reaches the terminal value:

$$\boxed{FV = PV\left(1 + \frac{r}{m}\right)^{mt}} \tag{2.9}$$

Solving for PV gives the companion discounting formula

$$\boxed{PV = FV\left(1 + \frac{r}{m}\right)^{-mt}.} \tag{2.10}$$

Example 1. The following table lists the value of \$100 after 1 year of compounding with frequency m and annual interest rate $r = 0.1$.

m (Compounding frequency)	$100(1 + 0.1/m)^m$ (Value of \$100 in 1 year)
1	110.00
2	110.25
4	110.38129
12	110.47131
52	110.50648
365	110.51558
10,000	110.51704
1,000,000	110.51709

□

Exercise 1. Find the amount you need to deposit now into an account paying the annual interest rate of 5% compounded monthly so that it becomes \$100,000 in 10 years.

Effective annual rate. If interest is reinvested with frequency m for t years at annual rate r then the effective annual rate r_m is the equivalent rate (the one that gives the same terminal value) with compounding frequency 1. Thus, r_m is found by solving the equation

$$PV(1 + r_m)^t = PV\left(1 + \frac{r}{m}\right)^{mt},$$

or

$$1 + r_m = \left(1 + \frac{r}{m}\right)^{m},$$

or

$$\boxed{r_m = \left(1 + \frac{r}{m}\right)^{m} - 1.} \tag{2.11}$$

Example 2. If the compounding frequency $m = 365$ and the annual interest rate $r = 0.1$ then the effective annual interest r_{365} is equal to:

$$r_{365} = \left(1 + \frac{r}{365}\right)^{365} - 1 \approx 0.10516$$

Exercise 2. Given an annual interest rate (nominal interest rate) of 0.04, find the effective annual rate when interest is compounded every minute. (ans. 0.0408107722)

Example 3: T-Bill Pricing

If you purchase a bond from a security dealer you would pay the ask price P_{ask}. If you sell a bond to this dealer you are paid the bid price P_{bid}. We can use the ask and bid yields to find the ask and bid prices for purchasing bonds. Brokers make money by purchasing at bid price, which is lower, and selling at the ask price, which is higher.

Purchase Date: December 19, 2008
Maturity Date: January 22, 2009
Bid yield on discount basis, $r_{dbid} = 0.080\%$
Ask yield on discount basis, $r_{dask} = 0.060\%$
Ask yield, $r = 0.061\%$

Calculate the ask price, P_{ask}, which is what you pay for $FV = \$1$ million T-Bill using the following formula.

Ask yield on a discount basis: The formula

$$r_{dask} \doteq \frac{d}{360} \frac{1 - P_{ask}}{1}$$

allows one to calculate the ask price. Note that the division by 360 simplified computations (in old days). The same is true for the division by 1 (FV) instead of P_{ask}.

Solution: Applying this formula with $d = 34$ (days) gives

$$0.00060 = \frac{34}{360} \frac{1 - P_{ask}}{1}$$
$$P_{ask} = 0.9936470588$$

Therefore, you pay **\$993,647.06** for a \$1 million T-Bill for 34 days.

Next, we calculate the ask yield, r_{ask}, which is the return $FV = \$1$ million T-Bill using the actual ask price.

Formula for actual ask yield:

$$r_{ask} = \frac{d}{365} \frac{1 - P_{ask}}{P_{ask}}$$

Note: We can also use corresponding equations to calculate the Bid Price, P_{bid}, and the Bid Price on a discount basis, P_{bid}.

Formula for bid yield on a discount basis:

$$r_{dbid} \doteq \frac{d}{360} \frac{1 - P_{bid}}{1}$$

Formula for actual bid yield:

$$r_{bid} = \frac{d}{365} \frac{1 - P_{bid}}{P_{bid}}$$

Exercise 3. In Example 3 find the bid price. What is the broker's profit.

2.3 Continuous Compounding

If the compounding frequency m becomes bigger and bigger then letting m go to infinity we obtain what is know as *continuous compounding*. In this case the terminal value PV is the limiting value of formula (2.9), that is

$$FV = \lim_{m\to\infty} PV\left(1+\frac{r}{m}\right)^{mt}.$$

Bringing PV outside the limit, making the substitution $n = m/r$ and noting that $m \to \infty$ is equivalent to $n \to \infty$ we have

$$\begin{aligned} FV &= PV \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n\cdot rt} \\ &= PV\left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\right]^{rt}. \end{aligned}$$

Now, using the well known **definition of** e

$$\boxed{e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}} \tag{2.12}$$

we obtain the continuous compounding formula:

$$\boxed{FV = PVe^{rt}} \tag{2.13}$$

Solving for PV we obtain the discounted value of a future amount

$$\boxed{PV = FVe^{-rt}.} \tag{2.14}$$

The following table indicates the convergence of the sequence defining e. In business language it lists the terminal values of \$1 after 1 year earning the annual interest rate of 100% with compounding frequency $n = 1, 2, 3, \ldots$:

n (Compounding frequency)	$(1+1/n)^n$ (Value of \$1 in 1 year)
1	2
2	2.25
4	2.44141
12	2.61304
24	2.66373
52	2.69260
365	2.71457
1,000	2.71692
10,000	2.71815
100,000	2.71827
1,000,000	2.71828

In mathematical terms, this sequence converges since it **increases and is bounded** from above. The table suggests that the upper bound is a little above 2.718. Indeed, it can be shown that $\left(1+\frac{1}{n}\right)^n < 3$ for all n. The limit is an important irrational number, which is denoted by the letter

e in honor of the great mathematician Leonhard Euler (1707-1783). Its approximation to 40 decimal places is:

$$e \approx 2.7182818284590452353602874713526624977572$$

Example 4. Suppose you invest \$10,000 in an account paying 4% annual interest, and you leave it there without adding or withdrawing anything. How much will you have at the end of 3 years if the interest is compounded continuously?

Solution. Using formula (2.13) with PV =10,000, $r = 0.04$ and $t = 3$ to get the amount

$$FV = 10,000e^{0.04 \cdot 3} \approx 11,274.97 \ \square$$

Exercise 4. How much money must be put into an account paying 5% annual interest, compounded continuously, in order to have \$10,000 at the end of 4 years? (ans. \$8,187.31)

Effective annual rate for continuous compounding. If r is the rate of interest in continuous compounding then the effective annual rate r_1 is found by solving the equation

$$PV(1 + r_1)^t = PVe^{rt}$$

or

$$1 + r_1 = e^r$$

or

$$\boxed{\text{effective annual rate} = r_1 = e^r - 1.} \tag{2.15}$$

More generally, to find the relation between an interest rate r_c with continuous compounding and a interest rate r_m with compounding frequency m we must solve the equation

$$PV\left(1 + \frac{r_m}{m}\right)^{mt} = PVe^{r_c t}$$

or

$$\boxed{\left(1 + \frac{r_m}{m}\right)^{m} = e^{r_c}} \tag{2.16}$$

Example 5. Given a nominal interest rate is 0.04, find the effective annual rate if interest is compounded continuously.

Solution. Using formula (2.15) we find

$$\text{effective annual rate} = e^{0.04} - 1 = 0.0408107742.$$

Exercise 5. Assume that a bank offers you a savings account with the annual interest rate of 5% compounded daily. What is the equivalent rate with continuous compounding?

2.4 Variable Annual Interest Rates

Next we discuss continuous compounding when the annual interest rate is variable, that is $r = r(t)$. For this let us assume that at $t = 0$ we are investing an amount A_0 which at any subsequent time t is earning annual interest rate $r(t)$ compounded continuously. If we denote by $A(t)$ the amount of our investment at time t and by $A(t+\Delta t)$ the amount of our investment at time $t+\Delta t$ then, assuming that $r(t)$ is continuous, we have

$$A(t+\Delta t) - A(t) \approx r(t)\cdot A(t)\cdot \Delta t.$$

Next, dividing by Δt gives

$$\frac{A(t+\Delta t) - A(t)}{\Delta t} \approx r(t)\cdot A(t).$$

Finally, letting Δt go to zero yields the first order differential equation (DE)

$$\frac{dA}{dt} = r(t)A. \tag{2.17}$$

Solving it with initial data $A(0) = A_0$ we obtain the formula

$$A(t) = A_0\, e^{\int_0^t r(s)ds},$$

which in the present/future value notation reads as follows:

$$\boxed{FV = PV\, e^{\int_0^t r(s)ds}.} \tag{2.18}$$

Exercise 6. Solve DE (2.17) to produce formula (2.18).

2.5 Coupon Bond

A holder of a coupon bond receives scheduled payments C_t at the end each period t (typically every 6 months). This situation is as follows:

r = Given interest rate

$PV(0)$ = Present value (at time $t = 0$) = "Discounted Value" of cash flow (because you are paying this amount for what will become a larger amount of money over time)

C_t = Amount of payment from coupon bond at a time t

For **Period 1:**

$$(1+r)PV(0) = C_1$$

Therefore,

$$PV(0) = \frac{C_1}{1+r}$$

For **Period N:**

$$(1+r)^N PV(0) = C_N$$

Therefore,

$$PV(0) = \frac{C_N}{(1+r)^N}$$

Summing up the discounted value of all coupons from the **N Periods** gives:

$$PV(0) = \frac{C_1}{(1+r)} + \frac{C_2}{(1+r)^2} + ... + \frac{C_N}{(1+r)^N} = \sum_{t=1}^{N} \frac{C_t}{(1+r)^t}$$

Example 6: Calculatin yield on U.S. government coupon note

Overview: The following example deals with a bond which originated on August 15, 2008, and accrued interest which was not yet paid until November 1, 2008. On November 1, the bond was sold to a new owner, and the first interest payment occurred on February 15, 2009. It finally matures on February 15, 2010. These dates are shown on the included timeline (see Figure 2). Other important characteristics of the bond are given throughout the example. The price paid on November 1, then, must take into account the value of the bond and the accrued, unpaid interest.

Date of Origination: February 15, 2008
Purchase Date: November 1, 2008
Maturity Date: February 15, 2010

Note: A Treasury note has maturity of $2, 3, 5$, and 10 years. Figure 1 records a quote for this U.S. government coupon bond on November 7, 2008. The table *Treasury Bonds, Notes and Bills* is from WSJ online. You can find this information by following the path at www.online.wsj.com. Home ⇒ Personal Finance ⇒ Investment ⇒ Data Center ⇒ Bond Rates ⇒ Treasury Quotes.

Treasury Bonds, Notes and Bills

Maturity	Coupon	Bid	Asked	Chg	Asked Yield
2010 Feb 15	6.500	106:17	106:18	+1	1.2734

Figure 1. Coupon bond maturing February 15, 2010

For each \$100 lent to the U. S. government:

Coupon Rate on annual basis: 6.5%
Coupon Rate every Six Months: $\$6.50/2 = \$3.25$

If you purchase a bond from a security dealer you would pay the ask price. If you sell a bond to this dealer you are paid the bid price:

Ask price: $106 + 18/32 = \$106.5625$
Bid price: $106 + 17/32 = \$106.53125$

The difference between the ask and bid would be the revenue for the dealer, which in this case is $\$3,125$ per million dollars of coupon bonds bought and sold.

The payments follow the time line

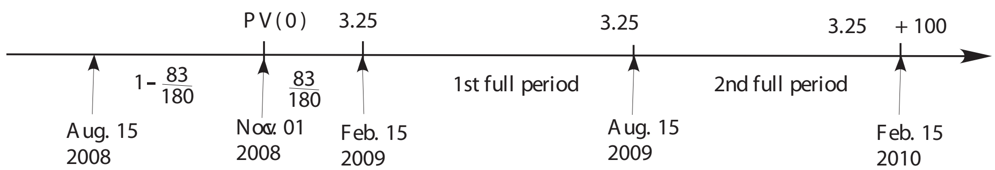

Figure 2

Using six month time period:

First cash payment: February 15, 2009
$t = 83$ days from November 1, 2008
$t = 83/180 = .5389$ of a period

In the last period of February 15, 2010 in addition to the coupon payment the original principal is paid back, say \$100.

Question: *Use the present value formula:*

$$PV(0) = \frac{C_1}{(1+r)^{t_1}} + \frac{C_2}{(1+r)^{t_2}} + \frac{C_3}{(1+r)^{t_3}}$$

to find the average annual rate of return (yield) y on this note.

As a result the present value of the cash flows is:

$$PV(y) = \frac{\$3.25}{(1+y/2)^{0.5389}} + \frac{\$3.25}{(1+y/2)^{1+.5389}} + \frac{\$3.25 + \$100}{(1+y/2)^{2+.5389}}.$$

Note: We divide the annual yield by 2 since the annual interest rate is paid every six months.

The purchaser of the bond pays the the ask price of $\$106\frac{18}{32}$. In addition, the purchaser pays the accrual of interest owed to the previous holder of the bond, since the payment on February 15, 2009 includes interest from August 15, 2008 to November 1, 2008. This accrual of interest is

$$\text{accrual} = (1 - 83/180) \times \$3.25 = \$1.498611111.$$

Price paid on November 1:

$$\begin{aligned} P_{Nov1} &= \text{ask price} + \text{accrual of interest} \\ &= 106.5625 + 1.498611111 \\ P_{Nov1} &= \$108.0611111 \end{aligned}$$

Yield to Maturity: The interest rate, y, which makes the present value of the bond payment, $PV(y)$ just equal to the the amount you pay.

$$\frac{\$3.25}{(1+y/2)^{0.5389}} + \frac{\$3.25}{(1+y/2)^{1+.5389}} + \frac{\$3.25 + \$100}{(1+y/2)^{2+.5389}} = \$108.0611111 \Rightarrow y = 0.01270191128 \text{ or } 1.27\%.$$

We found this value for y from the command fsolve in Maple. See the Maple file calc-lect-05&6.

Thus, the yield to maturity is the interest rate which assures the present value of the bond payments is equal to the amount paid for the bond.

Exercise 7. Find a U.S. government note maturing in 1211 February 15 and determine its yield to maturity like in the last example.

2.6 Annuities or Mortgages

Annuities, or mortgages, are used when an investor would like to borrow money and repay it over time in fixed payments. In this case, $d = \frac{1}{1+r}$ is constant.

Remember that:

$$FV = (1+r)PV \quad \Longleftrightarrow \quad PV = \frac{1}{1+r}FV$$

For the case of annuities in which we want to solve for the PV, we have

$$PV = dC,$$

where $d = \frac{1}{1+r}$ (constant for all payments) and C = amount of each payment.

For n payment periods:

$$PV = Cd + Cd^2 + ... + Cd^n$$

Therefore,

$$PVd = Cd^2 + Cd^3 + ... + d^{n+1}$$

Now, subtract the second equation from the first:

$$(1-d)PV = Cd(1-d^n)$$

or

$$PV = C\frac{d(1-d^n)}{1-d}$$

Substituting $d = \frac{1}{1+r}$,

$$PV = \frac{C}{r}\left[1 - (\frac{1}{1+r})^n\right]$$

Which also allows us to find the monthly payment, C, for an amount of money borrowed, PV:

$$C = \frac{rPV}{1-(\frac{1}{1+r})^n}$$

Example 7.

Find the monthly payment in the following situation:

PV = \$200,000

$r = 6\%/12$

$n = 360$ months (30 year mortgage)

Solution:

$$\begin{aligned} C &= \frac{rPV}{1-(\frac{1}{1+r})^n} \\ &= \frac{(\frac{0.06}{12})*(200,000)}{1-\left(\frac{1}{1+\frac{0.06}{12}}\right)^{360}} \\ C &= \$1,199.10 \end{aligned}$$

Exercise 8. Look up the average mortgage rate on a 30 year loan and determine amount you can borrow given you can afford $1,500 per month.

2.7 Introduction to Stock Pricing

Stocks do not come with the predictability or security of bonds:

1. Dividends are unknown until payment
2. There is no maturity date for a stock
3. In the case of bankruptcy, bond holders are paid off before the shareholders. The residual of the company's assets are divided among the shareholders.

Stock: Share of ownership in a company. There are two types of stocks:

Common: Pays a dividend payment each quarter. The company's profits are divided so that dividends per share equal reported profits divided by number of shares. Common shareholders have the right to vote in corporate elections.

Preferred: Pays a specified dividend before the common shareholders receive dividends but after the bondholders are paid. Preferred shareholders do not have voting rights.

Dividends: The share of a company's profits that each shareholder is entitled to (D_t).

Value: The value of a stock is considered the sum of the dividends paid for a particular stock before the company goes bankrupt. The value of the stock is denoted as S_0:

$$S_0 = \sum_{t=1}^{N} \frac{D_t}{(1+r_t)^t}$$

Exercise 9. Consider a simple world where a stock gives at the end of each year the same dividend $D = 10$ dollars. Also assume that the annual interest is 2% compounded annually. What is the price of this stock?

The time, N, at which the company goes bankrupt, the dividends at each time of payment, D_t, and the interest rate, r_t, and any time, t, are unknown. Therefore, the price of a stock is based on expectations of its future activity:

$$S_0 = E_0\left[\sum_{t=1}^{N} \frac{D_t}{(1+r_t)^t}\right]$$

The return on a stock is the profit a stockholder makes from selling a stock relative to the price at which it was purchased. The return, $R(t)$, is denoted as:

$$R(t) = \text{capital gain} + \text{dividend yield}$$
$$R(t) = \frac{S_t - S_0}{S_0} + \frac{\sum D_t}{S_0}$$

Note: The return on a stock, $R(t)$, is different from the interest rate, r_t, although each are dependent on the specific time, t.

People make decisions regarding the trading of stocks in advance of knowing the outcome of the stock based on expected returns:

$$E_0\left[R(t)\right] = E_0\left[\frac{S_t - S_0}{S_0} + \frac{\sum D_t}{S_0}\right]$$

There is a risk involved in purchasing stocks, that is not present while purchasing bonds. The expected value of bonds, $[(1+r)X_0]$, has a probability p=1 of occurring because the interest rate is predetermined. With a stock, however, there is the chance of the "up value" occurring or the "down value" occurring. For example, if the "up value," $S_1(H) = uS_0$, occurs with probability p, and the "down value," $S_1(T) = dS_0$, occurs with probability $1-p$, then the expected payoff of a stock is $p(uS_0) + (1-p)(dS_0)$. Later we shall go into more detail about the risks involved in purchasing stocks compared to purchasing bonds.

Example 8.

Find the possible rates of return for the following stock:

$S_0 = 50$

$S_1(H) = 100$

$S_1(T) = 25$

Dividends are 2% of S_0

Probability of $S_1(H) = p = 0.5$

Probability of $S_1(T) = 1 - p = 0.5$

For the higher stock price, the return is:

$$\begin{aligned} R(H) &= \frac{S_1(H) - S_0}{S_0} + \frac{D_t}{S_0} \\ &= \frac{100 - 50}{50} + 0.02 \\ R(H) &= 1.02 \text{ or } 102\% \end{aligned}$$

For the lower stock price, the return is:

$$\begin{aligned} R(T) &= \frac{S_1(T) - S_0}{S_0} + \frac{D_t}{S_0} \\ &= \frac{25 - 50}{50} + 0.02 \\ R(T) &= -0.48 \text{ or } -48\% \end{aligned}$$

The expected return is:

$$E_0[R(t)] = 0.5R(H) + 0.5R(T) = 0.5 \times 1.02 + 0.5 \times (-0.48) = .51 - .24 = 0.27.$$

2.8 Short Selling

Short Selling: An investor borrows a stock at t = 0 under the agreement that it will be returned at time $t = T$. The investor also pays a fee to the broker.

After borrowing the stock, the investor immediately sells it at price $S(0)$ with the expectation that the price will drop before time $t = T$. Once the prices drops, the investor buys back the stock for price $S(T)$ and returns it to the investor. The profit made would be equal to $S(0) - S(T)$ minus the fee to the broker and any transaction fees.

$$\text{Profit} = S(0) - S(T) - \text{Fee to broker for stock} - \text{Transaction fees}$$

Example 9. Suppose we look at the stock price in example 8 and you decide to short this stock since you think the probability of the low price is $1 - p = 0.9$. How much would you gain when the price goes up and down? What is your expected gain?

Solution. If the price is high your loss is

$$S_0 - S_1(H) = 50 - 100 = -50.$$

If the price is low your gain is

$$S_0 - S_1(T) = 50 - 25 = 25.$$

Your expected gain is

$$0.1 \times (-50) + 0.9 \times 25 = -5 + 22.5 = 17.5.$$

Exercise 10. Explain why you would have shorted Lehman Brothers stock in July 2007. What would have been the most beneficial term to maturity for you?

2.9 Continuous Income Streams

Suppose that at $t = 0$ we open an investment account with an initial amount A_0 and then we make continuous deposits at the rate of $S(t)$ per year. If the account pays an interest rate of r, compounded continuously, then we want to find the amount $A(t)$ in the account at any subsequent time t. We shall do this in two steps. First we shall model this situation with a differential equation. Then we shall solve it.

We begin the modeling by observe that $A(t)$ grows in two ways: first, because new money is continually being deposited and, second, because interest is continually being added. The first occurs at a rate of $A(t)$ per year, the second at a rate of r of the current balance. To translate this information into the form of a differential equation, we let Δt denote a very small amount of time, and we compare the balance at time t with the balance at time $t + \Delta t$. During that time interval the interest paid is approximately $rA(t)\Delta t$ (which represents money growing at a rate of r of the current balance $A(t)$ for a period of length Δt). The new money added is approximately

$S(t)\Delta t$ (which represents money being added at a constant rate of $S(t)$ for a period of length Δt). Therefore, the approximate change in the balance is given by the formula

$$A(t+\Delta t) - A(t) \approx rA(t)\Delta t + S(t)\Delta t,$$

and we can make the approximation as good as we want by taking Δt small enough. Dividing both sides by Δt, we get

$$\frac{A(t+\Delta t) - A(t)}{\Delta t} \approx rA(t) + S(t),$$

and by letting $\Delta t \to 0$ we obtain the **first order linear** differential equation

$$\frac{dA}{dt} = rA + S(t). \tag{2.19}$$

Our continuous stream investment problem has now been **modeled** by equation (2.19) and the initial condition

$$A(0) = A_0. \tag{2.20}$$

Solving initial value problem (2.19)–(2.20) gives the formula:

$$\boxed{A(t) = A_0 e^{rt} + \int_0^t S(\tau)e^{r(t-\tau)}d\tau} \tag{2.21}$$

Exercise 11. Solve initial value problem (2.19)– (2.20) to derive formula (2.21).

Exercise 12. Verify that initial value problem (2.19)– (2.20) models continuous income streams even in the case that $r = r(t)$ (variable) and solve it.

Exercise 13. Suppose you open a retirement account at age 30 with an initial amount of \$5,000 and then make continuous deposits at the rate of \$10,000 per year. If the account pays an interest rate of 7% , compounded continuously, what is the balance in the account at any given time? In particular, what is the balance at the retirement age of 65?

Exercise 14. Suppose that a home buyer plans to take a 15-year mortgage at an interest rate of 5% but cannot spend more than \$2,000 per month on payments. Here are two questions of obvious interest to the buyer.

(a) What is the maximum amount that she can afford to borrow?

(b) If the buyer borrows this maximum amount, how much total interest will she pay?

(For simplicity, assume that the interest is compounded continuously and that the payments are made continuously at the rate of \$24,000 per year. In practice, these assumptions may not be precisely satisfied, but they greatly simplify the problem and give close approximations to the actual solution.)

2.10 Review Exercises

1. If you would like to make a return of \$10,000 on an investment in one year, with a simple interest rate of $r = 0.05$ how much money should you plan to invest initially?

2. If you would like to make a return of \$10,000 on an investment in one year, that grows by a compounded quarterly interest rate of $r_Q = 0.05$, how much should you plan to invest initially?

3. Calculate the ask price, P_{ask} on a FV = \$1,000,000 T-bill maturing in 40 days, with an actual ask yield of $r_{ask} = 0.05\%$.

4. Given a nominal interest rate of 0.05, find the effective annual rate under the following circumstances:

(a) Interest is compounded daily

(b) Interest is compounded every hourly

(c) Interest is compounded continuously

5. Find the present value of a bond, $B(0)$, with a continuous cash flow $B(t) = \$5$ and interest rate, $r = 0.05$ held for 6 years.

6. The holder of a coupon bond receives semi-annual coupon payments of \$10 for 5 years while holding the bond at an interest rate of $r = 0.05$. What is the present value at the time of the bonds purchase of those coupon payments?

7. In order to purchase a new house, someone would like to borrow \$400,000 at an interest rate of $r = 5\%$ annually and make constant monthly payments to repay the mortgage. What is the value of the monthly payments on a 30-year mortgage?

Glossary of Terms[1]

- Borrower – An individual or organization who spends more than her current income.
- Bankruptcy – A legally declared inability or impairment of ability of an individual or organization to pay their creditors
- Corporate Bond – A bond issued by a corporation. The term is usually applied to longer-term debt instruments, generally with a maturity date falling at least a year after their issue date. (The term "commercial paper" is sometimes used for instruments with a shorter maturity.)
- Corporate Stock – A share, also referred to as equity, of stock means a share of ownership in a corporation (company).
- Debtor – An entity that owes a debt to someone else; the entity could be an individual, a firm, a government, or an organization. The counterparty of this arrangement is called a **creditor**.
- Dividends – Payments made by a corporation to its shareholder members. When a corporation earns a profit or surplus, that money can be put to two uses: it can either be re-invested in the business (called retained earnings), or it can be paid to the shareholders as a dividend.
- Future Value– The nominal future sum of money that a given sum of money is "worth" at a specified time in the future assuming a certain interest rate, or more generally, rate of return.
- Liquidity – An asset's ability to be easily converted through an act of buying or selling without causing a significant movement in the price and with minimum loss of value.
- Present Value – The value on a given date of a future payment or series of future payments, discounted to reflect the time value of money and other factors such as investment risk.

[1] Definitions come from Bodie, Kane, and Marcus (2008), and Wikipedia.

- Saver – An individual or organization who spends less than her current income.
- Security – A contract between a borrower and a saver, which specifies the terms of the deal.
- Short Selling or "shorting" – The practice of selling a financial instrument that the seller does not own at the time of the sale. Short selling is done with intent of later purchasing the financial instrument at a lower price.

References

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