Risk Equilibrium Binomial Model for Convertible Bonds Pricing

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Abstract

After analyzing of two types of convertible bonds pricing models, Structural Approach and Reduced-Form Approach, and indicating their respective limitations, a new model, *Risk Equilibrium Binomial Model*, is proposed by first introducing risk adjustment equation through continuous time, then discretizing it into binomial model, and finally combining with *Theory of Corporation Market Value Allocation*. The new model conquers the limitations of the above two classes of approach. In this approach, the two most innovative ideas are the definition of *Risk Burden Ratio* and *Theory of Corporation Market Value Allocation*, which are the cornerstone of this model. When the model is used to analyze convertible, some important conclusions are reached. Although in the paper the model is proposed to price convertible bonds, it can price other derivatives after very little modification. To prove correction and flexibility of this approach, in the final section, after simplified to price stock option, the model reaches the identical result as Black-Scholes formula's.

Key words: Convertible, Credit Risk, CRR, Risk Burden Ratio

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Introduction

Convertible bonds are hybrid instruments with characteristics of both debt and equity. They entitle their owner to receive coupons plus the return of principal at maturity, however, prior to maturity the holder may surrender them for a preset number of shares of stock. According to underlying asset, there are two classes of convertible pricing models: the model with underlying asset of the firm's asset value (Structural-Form Approach), and the model with underlying asset of the value of the issuer's common stock (Reduced-Form Approach).

Merton's model (1974), making use of the Black and Scholes (1973) option pricing model to value corporate liabilities, was the first modern model of default and is considered the first structural model. In Merton's model, the firm's capital structure is assumed to be composed by equity and a zero-coupon bond with maturity T and face value of D. The firm's equity is simply a European call option with maturity T and strike price D on the asset value and, therefore, the firm's debt value is just the asset value minus the equity value. This approach assumes a very simple and unrealistic capital structure and implies that default can only happen at the maturity of the zero-coupon bond. Another approach, within the structural framework, was introduced by Black and Cox (1976). In this approach defaults occur as soon as the firm's asset value falls below a certain threshold, in contrast to the Merton's approach, default can occur at any time.

McConnell and Schwartz's model (1986), which is the first reduced model, treats LYON (Liquid Yield Option Note) as a single security which is a function of the firm's stock price. The most egregious assumption of the model, as the authors describe, is the value of the LYON depends upon the value of the issuer's common stock rather than the total market value of the firm. This assumption precludes the possibility of bankruptcy since the dynamic process of stock price in the model keeps stock price above zero. So at maturity, the investor receives either the face value of the LYON or the conversion value, and there is no possibility of default. This means the reduced model overstates the value of the LYON. To patch the drawback of McConnell and Schwartz's model or take credit risk into consideration, several modifications have been added. In the research note of Goldman Sachs (1994), the credit adjusted rate, which is the weighted average between riskless rate and risky rate by weight of conversion probability, is introduced to discount the convertible value of the two nodes one period in the future. Tsiveriotis and Fernandes (1998) argue that "the equity upside has zero default risk since the issuer can always deliver its own stock whereas coupon and principal payments and any put provisions depend on the issuer's timely access to the required cash amounts, and thus introduce credit risk". To handle this, Tsiveriotis and Fernandes propose splitting convertible bonds into two components: a "cash-only part", which is subject to credit risk, and an "equity part", which is not. This leads to a pair of coupled partial differential equations that can be solved to value convertibles. A simple description of this model in the binomial context may be found in Hull (2003). Yigitbasioglu (2001) extends this framework by adding an interest rate factor and, in the case of cross-currency convertibles, a foreign exchange risk factor. Another method to take credit risk into the model is to exogenously introduce default process through a single jumping process which drives the equity value to zero and requires cash outlays to bondholders. The probability of default over the next short time interval is determined by a specified hazard rate. When default occurs, some portion of the bond (either its market value immediately prior to default, or its par value, or the market value of a default-free bond with the same terms) is assumed to be recovered. Authors who have used this approach in the convertible bond context include Davis and Lischka (1999), Takahashi et al. (2001), Hung and Wang (2002), and Andersen and Buffum (2003). As reduced models, the basic underlying state variable is still the firm's stock price (though some of the authors of these papers also consider additional factors such as stochastic interest rates or hazard rates).

These two classes of models have their respective advantages. When referring to realization, reduced models are more easy to implement than structural models. Structural models are harder to apply since the underlying asset, the firm's total asset, is usually unobservable, whereas, reduced models are constructed upon the underlying asset of stock price, the volatility of which can be estimated directly. However, when referring to credit risk, structural models are more plausible than reduced models, because reduced models do not consider the relation between default and firm value in an explicit manner, while structural models provide a link between the credit quality of a firm and the firm's economic and financial conditions. Thus, defaults are endogenously generated within the model instead of exogenously given as in the reduced approach.

For the advantages and disadvantages of these two classes of models, I propose a new model which can take credit risk into account as well as realization.

The structure of the paper is as follows: In the first section, continuous time model is used to deduce risk adjustment equation. In the second section, continuous time model is discretized into binomial model. In the third section, theory of corporate market value allocation is introduced to calculate convertible value through observed stock price in market. After introducing two important figures in section 4, different combinations of options of convertible bonds are valued and some interesting results are reached in section 5, 6 and 7.

1 Risk Equilibrium Pricing in Continuous Time

The main task of this section is to derive *Risk Adjustment Equation*, which is used in the next section to construct discrete binomial model. The assumption of this model is almost the same as Merton's model[5]:

(1) no transactions costs, taxes or problems with indivisibility of assets.

(2) sufficient investors with comparable wealth guarantee perfect liquidity of market, so each individual can buy or sell as much of an asset as he wants at the market price.

(3) there exists an exchange market for borrowing and lending at the same rate of interest.

(4) short-sales of all assets, with full use of the proceeds, is allowed.

(5) trading in assets takes place continuously in time.

(6) the Modigliani-Miller theorem that the value of the firm is invariant to its capital structure.

(7) the term structure is flat and known with certainty.

(8) the dynamics for the total market value of the firm A through time can be described by a *Geometric Brownian Motion* $dA/A = \mu dt + \sigma dw$ under

real world probability measurement (Ω, F, P). The drift rate μ and the volatility σ are known with certainty. w is a *Standard Wiener Process* under probability measurement (Ω, F, P). So there are no types of payouts by the firm during the lifetime of convertible bonds.

(9) the total market value of the firm A is composed of two parts: the total market value of convertible C and the total market value of stock S. So $A = C + S$, $C = c \cdot N_C$, $S = s \cdot N_S$, where c is the market price of convertible, N_C is the number of outstanding convertible, s is the market price of stock, and N_S is the number of shares of stock.

(10) the total market value of convertible is a function of the total market value of the firm and time, $C = C(A, t)$. It can be regarded as a special option of the total market value of the firm.

In the above assumptions, there are two main differences from Merton's assumptions. The first difference is the explicit stating that it is the total market value of the firm that follows Geometric Brownian Motion. In Merton's model, he did not manifest whether the value of the firm is book value or market value. The second difference is that dollar payouts by the firm are not concerned, which can be relaxed in the future.

Under the assumptions that the total market value of the firm follows a Geometric Brownian Motion,

$$dA = \mu A dt + \sigma A dw \tag{1}$$

and total market value of convertible is a function of A and t, we can derive the dynamics for C by Ito's Lemma,

$$dC = \left(C_t + C_A \mu A + \frac{1}{2} C_{AA} A^2 \sigma^2 \right) dt + C_A A \sigma dw \tag{2}$$

where subscript means first or second order derivative. Rewrite the above equation as

$$dC/C = \mu^C dt + \sigma^C dw$$

where

$$\begin{cases} \mu^C = \frac{C_t + C_A \mu A + \frac{1}{2} C_{AA} A^2 \sigma^2}{C} \\ \\ \sigma^C = \frac{C_A A}{C} \sigma \end{cases} \tag{3}$$

Following the Merton's derivation, we construct a three-security portfolio containing W_1 dollars of the total market value of the firm, W_2 dollars of convertible, and $W_3 \equiv -(W_1 + W_2)$ dollars of riskless debt with risk-free rate r. So the instantaneous dollar return of the portfolio is

$$\begin{aligned} dx &= W_1 \frac{dA}{A} + W_2 \frac{dC}{C} + W_3 r dt \\ &= \left[W_1 (\mu - r) + W_2 \left(\mu^C - r\right)\right] dt + \left(W_1 \sigma + W_2 \sigma^C\right) dw \end{aligned}$$

choosing W_1 and W_2 to make the coefficient ahead of dw to be zero, then the portfolio is riskless. And because the initial investment is zero ($W_1 + W_2 + W_3 \equiv 0$), according to the no arbitrage theory, the expected return of the portfolio should be zero.

$$\begin{cases} W_1 \sigma + W_2 \sigma^C = 0 & \text{no risk} \\ \\ W_1 (\mu - r) + W_2 \left(\mu^C - r\right) = 0 & \text{no arbitrage} \end{cases}$$

A nontrivial solution exists if and only if

$$\frac{\mu - r}{\sigma} = \frac{\mu^C - r}{\sigma^C} \triangleq \lambda$$

define *risk price* λ as the required risk premium per unit of risk. From above equation, if we get convertible risk σ^C, we can calculate its return as

$$\mu^C = r + \lambda \sigma^C \tag{4}$$

Now, the key problem is how to get σ^C.

By Ito's Lemma again, we get the dynamics of $\ln A$ and $\ln C$ as

$$\begin{cases} d\left(\ln A\right) = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dw \\ d\left(\ln C\right) = \left(\frac{C_t + C_A \mu A + \frac{1}{2}C_{AA}A^2\sigma^2}{C} - \frac{C_A^2 A^2 \sigma^2}{2C^2}\right)dt + \frac{C_A A}{C}\sigma dw \end{cases} \tag{5}$$

From the above equation for C, we define the coefficient before σdw as *Risk Burden Ratio for Convertible.*

$$a\left(A,t\right) = \frac{C_A A}{C} \tag{6}$$

which states how many percentage of risk convertible investors are taking with respect to the total risk of the firm. From equation 3, risk burden ratio for convertible a is the ratio of σ^C to σ. Or σ^C can be expressed through a.

$$\sigma^C = a \cdot \sigma \tag{7}$$

And it is not difficult to prove $0 \leq a \leq 1$.

We can not calculate a through equation 6, since we don't know C and C_A. However we can calculate a by another method which can be implemented easily in discrete model. From equation 5, the covariance of $d\left(\ln A\right)$ and $d\left(\ln C\right)$ is

$$\begin{aligned} cov\left(d\left(\ln A\right), d\left(\ln C\right)\right) &= cov\left(\sigma dw, a \cdot \sigma dw\right) \\ &= a\sigma^2 cov\left(dw, dw\right) \\ &= a\sigma^2 var\left(dw\right) \\ &= a\sigma^2 dt \end{aligned}$$

so a can be expressed through the covariance of $d(\ln A)$ and $d(\ln C)$.

$$a = \frac{cov\left(d\left(\ln A\right), d\left(\ln C\right)\right)}{\sigma^2 dt} \tag{8}$$

Rearrange equations 8, 7 and 4,

$$\begin{cases} a = \frac{cov(d(\ln A), d(\ln C))}{\sigma^2 dt} \\ \sigma^C = a \cdot \sigma \\ \mu^C = r + \lambda \sigma^C \end{cases}$$

from the above three equations, we can calculate convertible return μ^C through covariance of $d(\ln A)$ and $d(\ln C)$.

The risk burden ratio a is a percentage with respect to the standard deviation of the return of the total market value of the firm $\sqrt{var\left(d\left(\ln A\right)\right)} = \sigma$. When the firm faces bankruptcy (convertible investors take over the remaining assets of the firm $C = A$), or when the stock price is far above conversion price (convertible is acting as C_r shares of stock, so convertible, stock and total market value of the firm have the same return $d(\ln C) = d(\ln S) = d(\ln A)$), $a = 1$ means that the risk taken by convertible is the same as the risk of the total value of the firm; when stock price is far below conversion price and the probability of the bankruptcy of the firm is very small (convertible can only get constant income $var\left(d\left(\ln C\right)\right) = 0$), $a = 0$ means that convertible takes no risk and can be regarded as risk-free debt; at most times, convertible is in the middle of above situations, so convertible takes part of the risk of the total value of the firm $0 < a < 1$.

2 Risk Equilibrium Binomial Model

To simplify the description, we ignore conversion, call and put provisions at first, and in the following sections we will add those provisions step by step. Now the 'convertible' is actually a corporation bond ($C = B$). The parameters of the model are set as:

$N_S = 2e + 7$	number of shares of stock
$N_C = 1e + 6$	number of convertible
$s_0 = 10$	stock price observed in market
$c_0 = 786.069$	proposed convertible price [1]
$S_0 = 2e + 8$	total market value of stock
$C_0 = 7.86069e + 8$	proposed total value of convertible ($C0 = c0 \cdot N_C$)
$\mu = 0.06$	expected return of total market value of the firm
$\sigma = 0.5$	volatility of total market value of the firm
$r = 0.05$	risk-free interest rate
$\lambda = 0.02$	risk price
$F = 1000$	face value (par value) of convertible
$t = 0$	current time (unit: 1 year)
$T = 1$	expiration (unit: 1 year)
$L = 4$	numbers of steps of binomial model

Based on the above parameters, a binomial tree of A is constructed in a similar way as CRR model, but there is a slight difference. The CRR model is usually constructed in a risk neutral world, so the drift term is risk-free rate. Our model is in a real world, so the drift term should be the expected return of

[1] next section will explain it.

the total market value of the firm μ.

Let $\Delta t = (T - t)/L = 0.25$, and use the following equations

$$\begin{cases} u = \exp\left(\sigma\sqrt{\Delta t}\right) \ , \ d = 1/u \\ p_u = \frac{\exp(\mu \Delta t) - d}{u - d} \quad , \ p_d = 1 - p_u \end{cases}$$

to calculate binomial tree parameters as

$p_u = 0.467737$	probability of moving up
$p_d = 0.532263$	probability of moving down
$u = 1.28403$	ratio of moving up
$d = 0.778801$	ratio of moving down

Now the binomial tree is constructed as figure 1, in which there are 3 numbers at each node. The above number is A (unit: billion); the middle number is C (unit: billion); the below number is risk burden ratio a (at the nodes at expiration a is undefined).

At expiration, total convertible value is a function of total firm value.

$$C_T = \min\left(A_T, F \cdot N_C\right)$$

If total firm value is less than the product of face value and numbers of convertible, then the firm is bankrupt and convertible investors take over all the asset of the firm, else convertible investors get constant face value. From the above parameters, $F \cdot N_C = 1e+9$, so when A_T is larger than $1e+9$, C_T is $1e+9$ (at nodes E1 & E2), and when A_T is less than $1e+9$, C_T is A_T (at nodes E3, E4 & E5).

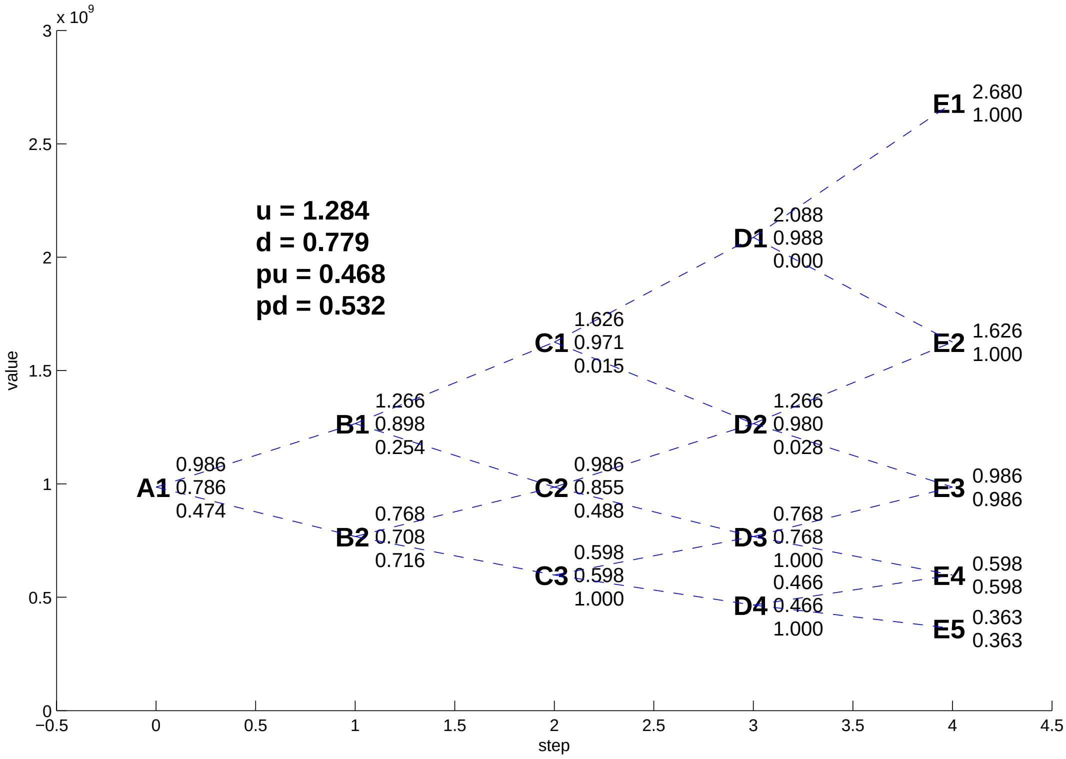

Fig. 1.

Once we get convertible value at expiration, we can roll back in time down the tree. At first, a single period binomial model composed of nodes D2, E2 & E3 is used as an example to explain the process. When we stand at node D2 (current time $t = 0.75$), there are two possible states in the close future (E2 & E3 at time $t + \Delta t = 1$). The probability of E2 is $p_u = 0.468$, the total market value of the firm is $A_u = 1.626$, and the total convertible value is $C_u = 1$. The probability of node E3 is $p_d = 0.532$, the total market value of the firm and the total convertible value are both the same $A_d = C_d = 0.986$. Now we can calculate the covariance of $\ln A_{t+\Delta t}$ and $\ln C_{t+\Delta t}$ at D2 through following process.

$$\begin{aligned} E\left(\ln A_{t+\Delta t}\left|\phi_t\right.\right) &= p_u \ln A_u + p_d \ln A_d \\ E\left(\ln C_{t+\Delta t}\left|\phi_t\right.\right) &= p_u \ln C_u + p_d \ln C_d \\ cov\left(\ln A_{t+\Delta t}, \ln C_{t+\Delta t}\left|\phi_t\right.\right) &= p_u\left[\ln A_u - E\left(\ln A_{t+\Delta t}\left|\phi_t\right.\right)\right]\left[\ln C_u - E\left(\ln C_{t+\Delta t}\left|\phi_t\right.\right)\right] \\ &\quad + p_d\left[\ln A_d - E\left(\ln A_{t+\Delta t}\left|\phi_t\right.\right)\right]\left[\ln C_d - E\left(\ln C_{t+\Delta t}\left|\phi_t\right.\right)\right] \end{aligned}$$

get $cov\left(\ln A_{t+\Delta t}, \ln C_{t+\Delta t}\left|\phi_t\right.\right) \approx 0.001756$. Substitute it into discrete version of equation 8.

$$a \approx \frac{cov\left(\Delta\left(\ln A\right), \Delta\left(\ln C\right)\left|\phi_t\right.\right)}{\sigma^2 \Delta t} = \frac{cov\left(\ln A_{t+\Delta t}, \ln C_{t+\Delta t}\left|\phi_t\right.\right)}{\sigma^2 \Delta t}$$

while $\sigma^2 \Delta t = 0.5^2 \times 0.25 = 0.0625$, so the risk burden ratio at D2 is $a \approx 0.001756/0.0625 \approx 0.028$ as the third number at node D2. From equation 7, the volatility of convertible at D2 is $\sigma^C = a \cdot \sigma \approx 0.028 \times 0.5 \approx 0.014$. Using equation 4, the expected return of convertible is $\mu^C = r + \lambda\sigma^C \approx 0.05 + 0.02 \times 0.014 \approx 0.05028$, which is the interest rate at D2 to discount convertible. So the total convertible value can be expressed as the discounted version of expected value at close future based on current information set.

$$C_t = \exp\left(-\mu^C \Delta t\right) E\left(C_{t+\Delta t}\left|\phi_t\right.\right) = \exp\left(-\mu^C \Delta t\right)\left(p_u C_u + p_d C_d\right)$$

Now we get $C_t \approx 0.980$ as the second number at node D2.

By same process, at node D1, since the two succeeding child nodes E1 & E2 have the same convertible value $C_{t+\Delta t} = 1.0$, so the covariance, the risk burden ratio and the volatility of convertible are all zero. At this time, convertible is riskless, and should be discount with risk-free interest rate $\mu^C = r$.

At nodes D3 and D4, since their succeeding child nodes are all in situation of bankruptcy ($C_{t+\Delta t} = A_{t+\Delta t}$), the covariance is $cov\left(\ln A_{t+\Delta t}, \ln C_{t+\Delta t}\left|\phi_t\right.\right) = \sigma^2 \Delta t$, and risk burden ratio is $a = 1$. So the volatility of convertible is the same as the volatility of the firm ($\sigma^C = \sigma$). The interest rate which is used to discount convertible should be the expected return of firm ($\mu^C = \mu$).

After we complete all the nodes at time $t = 0.75$, we can roll back to time $t = 0.5$ through the same method. And finally the node A1 is reached and total convertible value is 0.786 billion.

3 Theory of Corporation Market Value Allocation

Readers may feel suspicious about "proposed convertible price" in the above parameters set, and think there is a vicious circle. What we want to get is convertible price c_0, but we set it as known by proposing. To answer this question, we first establish *Theory of Corporation Market Value Allocation* (CMVA).

Merton, who first introduced the risk structure of interest rates to price corporation debt, assumed that the dynamics of the total market value of the firm is a Geometric Brownian Motion which was not affected by the capital structure of the firm (MM theory), and convertible was regarded as a derivative dependent on the total market value of the firm.

The above assumption can be viewed from another angle. The total market value of the firm follows a Geometric Brownian Motion, and at any time it is exogenous to any allocation between convertible investors and stock investors. The more value the convertible investors allocate, the less the stock investors allocate, and vice versa. The key problem is how the two types of investors can reach balance, which is the result of the competition between two parties of investors and is the result of rational investors pursuing return and avoiding risk. This is the Theory of Corporation Market Value Allocation, which is another perspective view of MM theory. MM theory believes capital structure has no effect on total market value of the firm, while CMVA treats the total market value of the firm as a exogenous number which is divided between different types of investors.

Combining CMVA and risk equilibrium binomial model, we will explain how the total market value of the firm is allocated between convertible investors and stock investors.

Actually, the binomial model tells us that total convertible value is a function of total firm value.

$$C = f(X_A) \ , \ X_A \in [0, +\infty)$$

By CMVA, as a claim on the residual of firm's asset, stock get

$$S = X_A - C = X_A - f(X_A) \tag{9}$$

Although the total market value of the firm is invisible, total stock value S^* can be observed. After substituting observed value S^* into above equation to solve implicit function of X_A to get A^*, and subtracting S^* from A^*, we get total convertible value.

$$C^* = A^* - S^*$$

Then we get convertible price $c^* = C^*/N_C$.

The above process can be described in another way to facilitate the program implementation. After the total stock value is observed and let

$$X_A = S^* + X_C \ , \ X_C \in [0, +\infty)$$

the equation 9 can be rewritten as

$$X_C = f(S^* + X_C)$$

or be expressed as the function of convertible price $x_c = X_C/N_C$

$$x_c = f\left(S^* + x_c N_C\right)/N_C$$

here, x_c is the "proposed convertible price" in the above parameters set. In the domain of definition $x_c \in [0, +\infty)$, solving the above implicit function we get convertible price c^*.

4 Delta Hedging and Risk Burden Ratio for Stock

From the above analysis, we know that risk equilibrium binomial model actually describes the competition process between convertible investors and stock investors, and finally established a mapping from A to C ($A \xrightarrow{f(A)} C$), while CMVA tells us how to use observed stock price to derive the total market value of the firm which can be supported at current stock price, and finally get convertible price. Now the whole model is completely established through above two sections.

Before further analysis of this model on convertible pricing, two important figures should be introduced: Delta Hedging Δ and Risk Burden Ratio of Stock a_s.

The Greek letter Δ is defined as the rate of change of the convertible price with respect to the stock price. It is the slope of the curve which relates the convertible price to stock price. It is number of stock shares that convertible investors should short to hedge the risk brought by one share of convertible. There are two method to calculate Δ: numeric method which differentiates convertible price over a range of stock price, and formula method whose de-

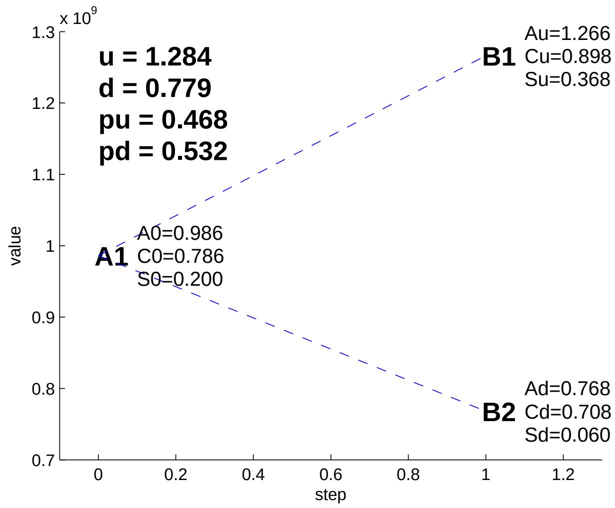

Fig. 2.

duction is described as follows. From definition

$$\Delta \triangleq \frac{\partial c}{\partial s} = \frac{\partial \left(C/N_C\right)}{\partial \left(S/N_S\right)} = \frac{N_S}{N_C}\frac{\partial C}{\partial S} = \frac{N_S}{N_C}\frac{\partial C/\partial A}{\partial S/\partial A} = \frac{N_S}{N_C}\frac{\partial C/\partial A}{\partial \left(A-C\right)/\partial A} = \frac{N_S}{N_C}\frac{C_A}{1-C_A}$$

from 6 we get $C_A = aC/A$, substitute into above equation

$$\Delta = \frac{N_S}{N_C}\frac{aC}{A-aC}$$

this is the formula to calculate Δ. Since in my program I get almost the same result from the two methods, formula method will be used in the following section by default.

As a more accurate figure to a corporation's leverage of debt, the risk burden ratio for stock measures how many times of corporation risk the stock investors take because of debts. There are also two methods to calculate it: numeric method and formula method.

To explain the numeric method, let's look at the following graph, which is the first step of the tree in figure 1.

The total stock value at nodes B1 and B2 are $S_u = A_u - C_u$ and $S_d = A_d - C_d$

respectively. As the process to calculate the risk burden ratio for convertible, we can express the risk burden ratio for stock by covariance of $d\left(\ln A_{t+\Delta t}\right)$ and $d\left(\ln S_{t+\Delta t}\right)$.

$$\begin{aligned}
E\left(\ln A_{t+\Delta t}\left|\phi_t\right.\right) &= p_u \ln A_u + p_d \ln A_d \\
E\left(\ln S_{t+\Delta t}\left|\phi_t\right.\right) &= p_u \ln S_u + p_d \ln S_d \\
cov\left(\ln A_{t+\Delta t}, \ln S_{t+\Delta t}\left|\phi_t\right.\right) &= p_u \left[\ln A_u - E\left(\ln A_{t+\Delta t}\left|\phi_t\right.\right)\right]\left[\ln S_u - E\left(\ln S_{t+\Delta t}\left|\phi_t\right.\right)\right] \\
&\quad + p_d \left[\ln A_d - E\left(\ln A_{t+\Delta t}\left|\phi_t\right.\right)\right]\left[\ln S_d - E\left(\ln S_{t+\Delta t}\left|\phi_t\right.\right)\right]
\end{aligned}$$

and divided by $\sigma^2 \Delta t$, then we get

$$a_s = \frac{cov\left(\ln A_{t+\Delta t}, \ln S_{t+\Delta t}\left|\phi_t\right.\right)}{\sigma^2 \Delta t}$$

Formula method to calculate risk burden ratio for stock is deduced as following. From equation 1 and 2, the dynamics of S is

$$dS = d\left(A - C\right) = \left(\mu A\left(1 - C_A\right) - C_t - \frac{1}{2} C_{AA} A^2 \sigma^2\right) dt + \sigma A\left(1 - C_A\right) dw$$

By Ito's Lemma, the dynamics of $\ln S$ is

$$d\left(\ln S\right) = \left(\frac{\mu A\left(1 - C_A\right) - C_t - \frac{1}{2} C_{AA} A^2 \sigma^2}{S} - \frac{1}{2}\left(\frac{\sigma A\left(1 - C_A\right)}{S}\right)^2\right) dt + \sigma \frac{A\left(1 - C_A\right)}{S} dw$$

The coefficient ahead of Standard Wiener Process dw is the product of the volatility of total firm value and the risk burden ratio for stock (σa_s), so

$$a_s = \frac{A\left(1 - C_A\right)}{S}$$

simplify above equation with notices that $S = A - C$ and $a = AC_A/C$,

$$a_s = \frac{A\left(1 - C_A\right)}{S} = \frac{A\left(1 - C_A\right)}{A - C} = \frac{A/C - AC_A/C}{A/C - 1} = \frac{A/C - a}{A/C - 1}$$

This is the formula to calculate a_s through A, C and a. Since in my program I get almost the same result from the two methods, formula method will be used in the following section by default.

Intuitively, if stock price is relatively low, then the probability of default is not trivial, and the stock investors take too much risk. The risk burden ratio for stock is very high at this moment. If stock price is relatively high, then the default is almost impossible, and the risk of the firm is almost taken by the stock investors. Bear this figure in mind, I will show you how perfect the model captures this character in the following sections.

5 Common Corporation Bonds

Before we consider the options of conversion, call and put, making the analysis of ordinary corporation bonds as a benchmark for comparison is necessary. For simplicity, we assume that corporation bonds were zero-coupon bonds, and in the bond's lifetime no dividend would be issued. The parameters are set as following. Although most of the parameters do not accord with reality, the reason is to let the result present the characters of the model.

$N_S = 5e + 6$	number of shares of stock
$N_C = 1e + 6$	number of convertible
$\mu = 0.06$	expected return of total market value of the firm
$\sigma = 0.07$	volatility of total market value of the firm
$r = 0.025$	risk-free interest rate
$F = 1000$	face value (par value) of convertible
$t = 0$	current time (unit: 1 year)
$T = 5$	expiration (unit: 1 year)
$L = 128$	numbers of steps of binomial model

We calculate bond price, risk burden ratio for bond, delta and risk burden ratio for stock within stock price range from 1 to 100.

From figure 3, when stock price is low, the probability of default is high. Suffering from such high credit risk, investors ask more risk premium. So bond price falls to about 720 and risk burden ratio for bond rises to about 0.9. The number 45 in Delta graph tells us that the risk of bond is the same as the risk taken by 45 shares of stock. And the number 14 in the final graph about risk burden ratio for stock tells us that 14 times the risk with respect to corporation risk are taken by stock investors. When stock price rises to almost 100, the probability of default is very small and can be ignored. So the bond can be treated as riskless bond, bond price reaches a plateau, risk burden ratio for bond is nearly zero, and delta is also about zero. The most interesting part

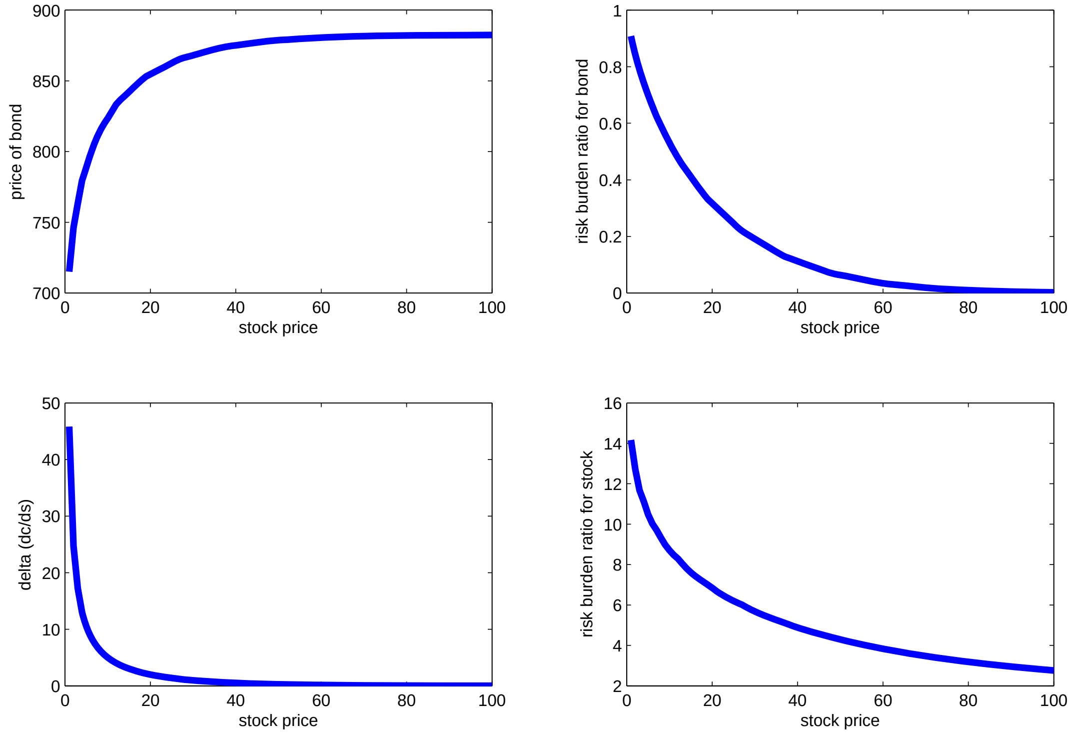

Fig. 3.

is that risk burden ratio for stock converges to the ratio of total firm value to total stock value ($A/S \approx 2.76$). It means that the whole risk of the firm is now only taken by the stock investors and bond investors take none risk.

6 Corporation Bonds with Conversion

Conversion provision is composed of two components: *conversion price* and *conversion period*. Conversion price is an execution price at which convertible can be transformed into stock. Another concept, *conversion ratio*, is defined as the numbers of stocks one convertible can convert. Conversion ratio equals to face value over conversion price.

$$C_r = F/K$$

Conversion period is an interval in which convertible holders can execute conversion. It usually starts from half years after issue and continues until expiration.

Definition 1 *Total hold value H is the total value of convertible when the investors and issuer do not execute any options. At expiration it equals total face value plus coupon (zero-bonds have no coupon). Before expiration H is the discounted value of expectation of total convertible value in the next step.*

$$H = \begin{cases} F \cdot N_C & , t = T \\ \exp\left(-\mu^C \Delta t\right) E\left(C_{t+\Delta t} \left| \phi_t \right.\right) & , t < T \end{cases}$$

Definition 2 *Total conversion value X is the total market value of new converted shares which the convertible investors get from converting their convertibles. It's the value that concerns the dilute effect.*

$$X = \frac{A \cdot N_C \cdot C_r}{N_S + N_C \cdot C_r}$$

In the process of rolling back in the binomial tree, if the nodes are in the conversion period, rational convertible investors will maximize their asset by choosing the larger of hold value and conversion value. If the total market value of the firm can not grant the reasonable demands of convertible investors, the firm suffers from bankruptcy and convertible investors take over the whole corporation's asset.

$$C = \min\left[\max\left(H, X\right), A\right]$$

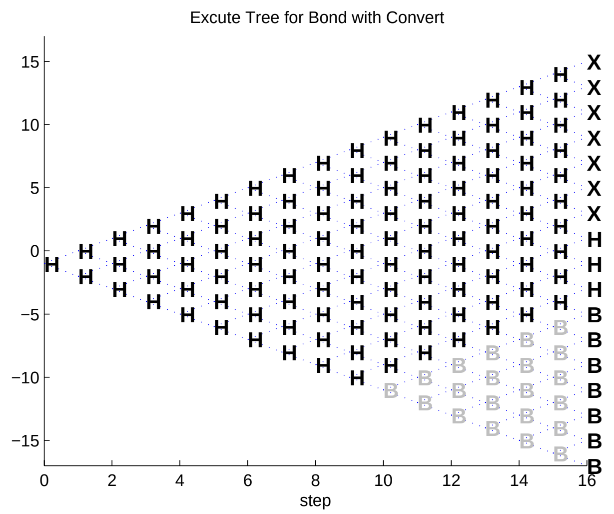

Fig. 4.

To illustrate the behaviors of investors and issuer and facilitate future discussion, let's define 5 situations, which are expressed in capital letters.

(1) H: convertible will be held for one more period without any options being executed.
(2) X: investors execute conversion provision to convert to stock.
(3) C: issuer exercises call.
(4) P: investors exercises put
(5) B: issuer suffers from bankruptcy.

We use following '16-steps execution tree' to show you the different behaviors investors and issuer choose at different situations. Since currently we are only concerned with the conversion provision, only behavior H, X and B are in the graph.

Through scrutiny, two points should be noted.

(1) Investors choose conversion only at expiration. It coincides with the char-

acter of American option that is not executed before expiration [2].

(2) Only at expiration when investors reclaim debts from corporation, bankruptcy can happen. Let's focus on the grey range of B in above graph. In this range, hold value H equals to total market value of the firm A, the firm has already stepped into the predicament of bankruptcy. So it can be labeled as 'H'.

Additional to the parameters of the last section, conversion parameters are set as

$K = 50$	conversion price
$ConversionPeriod = 0.5\tilde{}5$	conversion period

We calculate bond price, risk burden ratio for bond, delta and risk burden ratio for stock within stock price range from 1 to 100.

When stock price is far below conversion price 50, convertible price, risk burden ratio for convertible, delta and risk burden ratio for stock are almost the same as ordinary bond in the last section. It means that conversion provision has little effect on convertible price when stock price is far below conversion price and conversion in future is almost impossible. When stock price is above 70, convertible price rises along a straight line with slope and/or delta equal to conversion ratio $C_r = F/K = 20$. Convertible has the same risk as 20 shares of stock. And convertible and stock take the equivalent of risk, so $a = a_s = 1$. When stock price is near 20, convertible price reaches a flat roof, and risk burden ratio for convertible and delta reach their minimum.

[2] In real market, conversion before expiration may be caused by the relatively high exchange cost or poor liquidity with respect to stock.

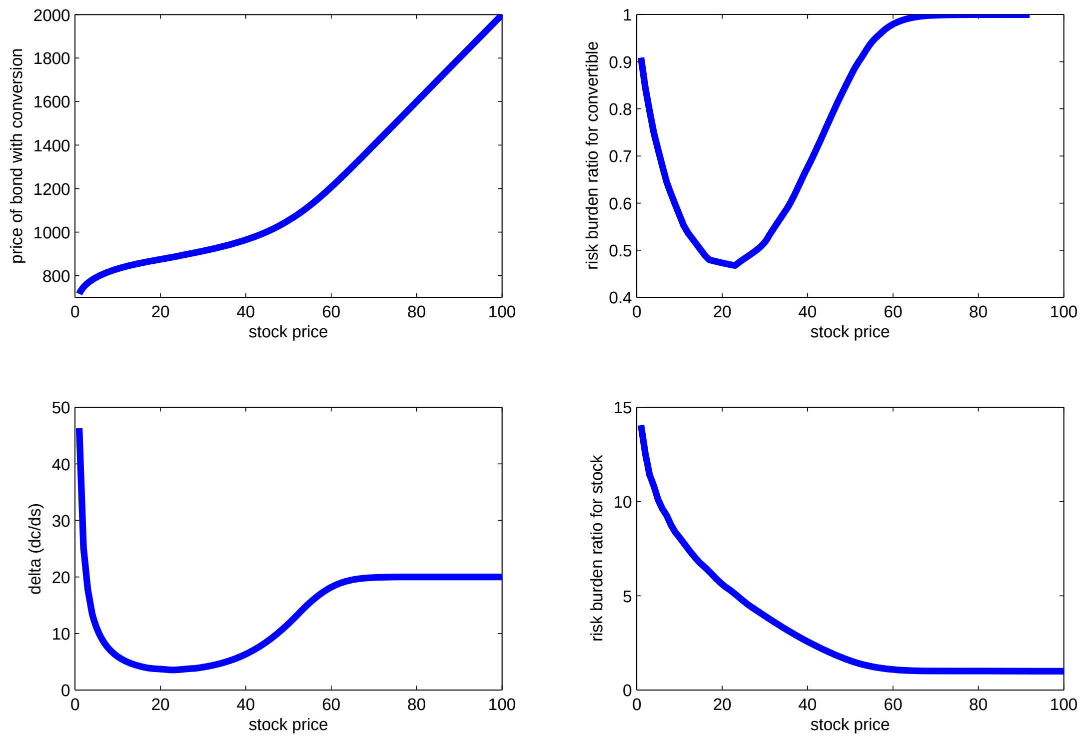

Fig. 5.

7 Corporation Bonds with Conversion, Call and Put

Since the effective range of call and put provision are different, in that call works when stock price is high whereas put works when it low, in this section call and put will be concerned at the same time.

Call provision is composed of two components: *call condition* and *call price*.

(1) Call condition usually stipulates a period from 12 months after issue day to expiration, in which if stock price keeps above 130% of conversion price for several trading days issuer has the right to redeem convertibles.

(2) Call price is a certain proportion of face value. In our example, it is set to 105%. So call price equals $105\% \cdot F = 1050$.

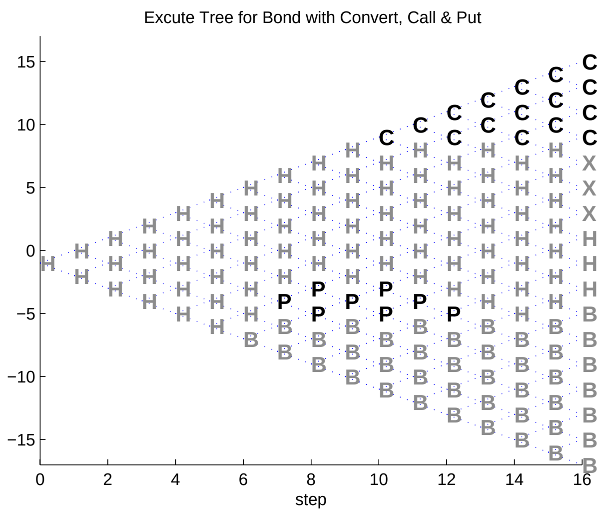

Fig. 6.

Put provision is also composed of two components: *put condition* and *put price*.

(1) Put condition usually stipulates a period from 24 months after issue day to expiration, in which convertible holders have the right to execute put option.

(2) Put price is a certain proportion of face value. In our example, it is set to 95%. So when the convertible price is below $95\% \cdot F = 950$, convertible holder will put convertible back to issuer for this price.

To illustrate the operation, the following '16-steps execution tree' is used to show you the different behaviors investors and issuer choose at different situations. The five situations, H, X, C, P and B, all appear in the execution tree.

Note that convertible price is capped by call provision. However most of convertible have a period of 30 days between the day issuer publishes redemption notice and the day he executes redemption. In this period, convertible holders have the right to choose conversion or accept issuer's redemption. Put provision sets a price floor to convertible. But when the firm's asset can not satisfy convertible holders' demand, bankruptcy happens.

Additional to the parameters of the last section, the parameters for call and put are set

$CallPeriod = 1\tilde{}5$	call period from 1 year after issue day to expiration
$CallCondition = 130\%$	percentage of face value
$CallPrice = 1050$	call price
$PutPeriod = 2\tilde{}5$	call period from 2 years after issue day to expiration
$PutPrice = 950$	put price

We calculate convertible price, risk burden ratio for convertible, delta and risk burden ratio for stock within stock price range from 1 to 100 as the figure 7 describes.

After careful comparison of figure 5 and figure 7, we find:

(1) In the model with call provision, if corporate publishes redemption notice, convertible holders usually have to choose conversion, that results in the immediate vanishing of some residual time value of convertible. Convertible price equals 20 shares of stock. So call provision accelerates the speed the convertible price enters into the range with slope 20.

(2) When stock price is low, put provision cushions this downside effect. The maximum of delta is about 35, below the delta of convertible without put. Since convertible has protection of put provision, a lot of risk is taken by

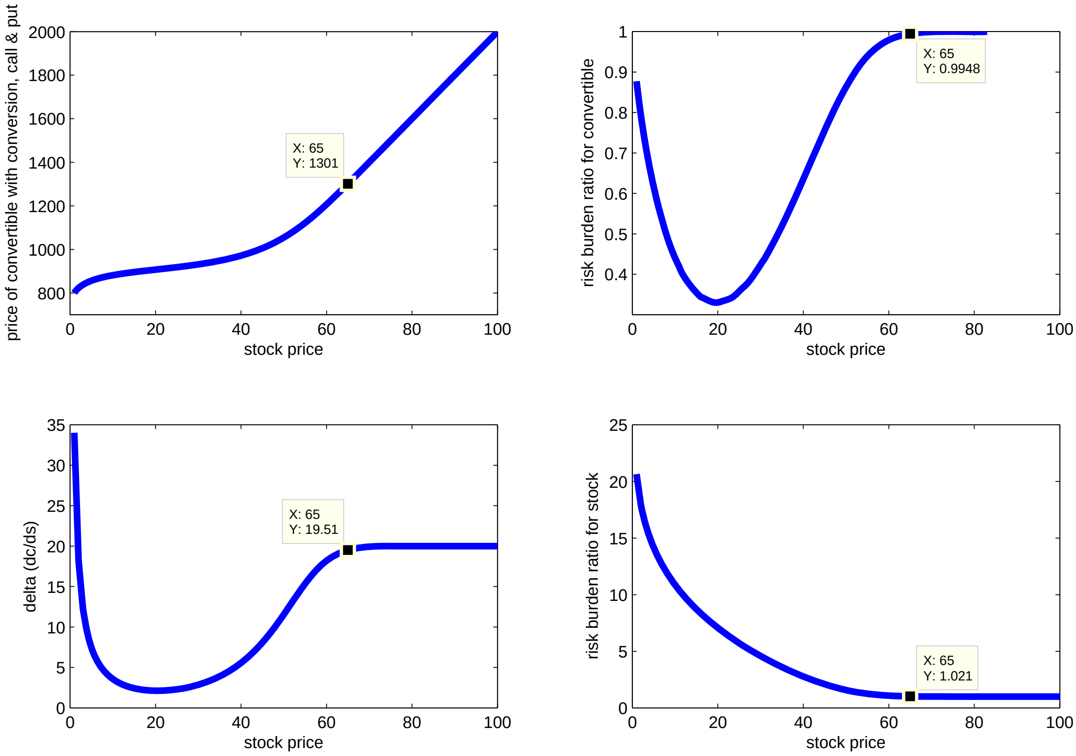

Fig. 7.

stock investors. The risk burden ratio for stock is above 20, compared with 14 in the model without put.

8 Comparison of Risk Equilibrium Binomial and Black-Scholes

Maybe, readers still suspect the correctness of this approach. To further confirm you, after several slight modifications the approach is used to price stock option, and the identical results with BS formula are reached.

(1) Since BS formula assumes stock price follows a Geometric Brownian Motion with constant drift and volatility term, but Risk equilibrium Binomial model assumes it is the firm's total market value that follows the

Geometric Brownian Motion, some parameters should be set to ensure the two models have the identical assumptions, then they are comparable. So we assume there are a corporate whose capital structure is composed of only shares. The number of convertible is set to $N_C = 0$.

(2) Stock option worth nothing if stock price is not larger than its strike price at expiration. But convertible bond is different, for it can draw back it principal (Face Value) even if it do not execute at expiration. So the face value is set to $F = 0$.

(3) One stock option has the right to call or put one share of stock at strike price. But convertible bond's conversion ratio is $C_r = F/K$. So the conversion ratio is set to $C_r = 1$.

(4) Finally, change the terminal condition to $c(s, T) = \max(s - K, 0)$.

After the above 4 modifications, the Risk equilibrium Binomial model can be used to price stock option.

Consider European stock options that expire in three months with an exercise price of \$95. Assume that the underlying stock pays no dividend, trades at \$100, and has a volatility of 50% per annum. The risk-free rate is 10% per annum. Using this data returns call and put prices of \$13.70 and \$6.35 by Black-Scholes formula, respectively.[3]

To different drift term μ, graphs are plotted to show the option price from the Risk Equilibrium Binomial model with respect to increasing number of steps (or decreasing value of step size).

[3] The example is from Hull, John C., Options, Futures, and Other Derivatives, Prentice Hall, 5th edition, 2003.

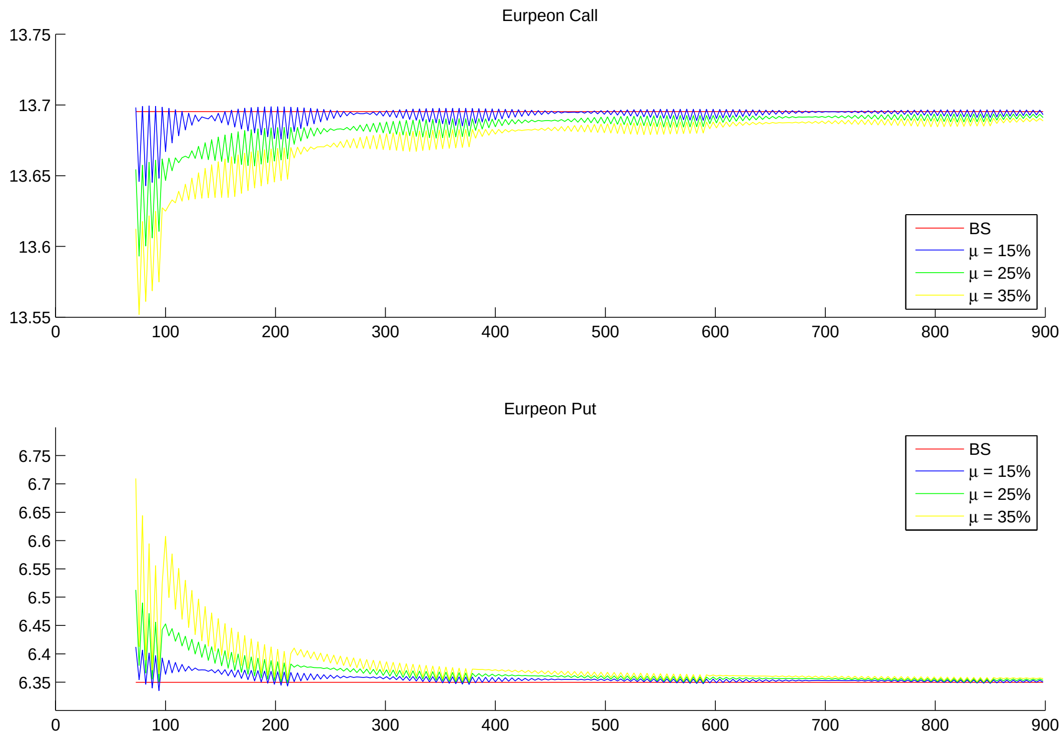

Fig. 8.

From the graphs, with the increasing number of steps, the results for call and put options converge to the values from BS formula. When the number of steps is large enough, the three lines with different drift term converge to 13.6953 for call option and 6.3497 for put option. Even though the Risk Equilibrium Binomial model is constructed in the real probability space (real world), it reaches the identical result with BS formula which can be derived from risk neutral world, in which the pricing process has no relation with drift term μ. It is very interesting to note that drift term do not appear in BS formula.

This result is exciting. For it proves the correctness of the approach as well as its flexibility to adapt different kinds of derivatives.

9 Conclusion

This model is composed of two parts: Risk Equilibrium Binomial Model and Theory of Corporation Market Value Allocation. The former part actually describes the competition process between equity investors and convertible investors. It is a process whereby two types of investors divide the firm's assets under fully apprehension about the competitors' possible behaviors in future. The latter part, however, tells us how to use the observed stock price to deduce convertible price. Adding the two parts together is the combination of "*Game*" and "*Equilibrium*".

The model is constructed upon the dynamic process of the firm's assets. So it belongs to structural models. The parameters, such as volatility and drift term of firm's assets, can be estimated through the historical total market values of the firm, which are the sum of total stock value and total convertible value. When we want to calculate rational convertible price, only current observed stock price is used, along the above estimated parameters and the firm's capital structure parameters.

Differing from Goldman Sachs (1994) and some common binomial model, the model is not constructed in the risk neutral world but in the real world, in that the interest rate μ, which is the rate with credit spread, is used instead of risk free interest rate r when binomial tree is built. In the process of rolling back through binomial tree, credit spreads at each nodes are calculated through risk burden ratio instead of simply weighted averaging between riskless rate and risky rate by weight of conversion probability in Goldman Sachs (1994) model or simply discounting "cash-only part" and "equity part" with risky rate and riskless rate respectively in Tsiveriotis and Fernandes (1998) model. The method to calculate credit spread through risk burden ratio has a solid theoretical foundation since it is deduced from continuous time model. It is more valuable in that risk burden ratio is a better index than the equity-to-debt ratio, the leverage of capital structure, and it can be wildly used to price different kinds of financial instruments as call and put warrant, mortgage bond, forward contract, future contract, call and put option and even the common stock as the paper describes. If a instrument is written by third part instead of the issuer who issues the underlying assets of the instrument (the net holding volume of the instrument is zero), the model can be used by simply setting N_C to zero, that means the value of the instrument has no effect on the total market value of the firm.

Based on the framework of the model, some extensions to the model can be introduced. For example, the coupons of convertible, the dividends of stock, the term structure of interest rate should be concerned; the tax and bankruptcy cost should be taken into consideration to relaxed the assumption that the Modigliani-Miller Theorem holds; the simple capital structure of equity and convertible should be relax to take in corporate bond, preferred stock, warrant and/or other instruments. A further research for above extensions is needed.

References

[1] Bardhan, I., Bergier, A., Derman, E., et al. Valuing Convertible Bonds as Derivatives. Goldman Sachs Quantitative Strategies Research Notes, 1994, July.

[2] Brennan, M.J. and Schwartz, E.S.. Conditional Predictions of Bond Prices and

Returns. The Journal of Finance, 1980b, 35(2):405-417.

[3] Jarrow R A, Turnbull S M. Pricing Derivative Securities on Financial Securities Subject to Credit Risk. Journal of Finance, 1995, 50: 53–86.

[4] McConnell, J.J., Schwartz, E.S.. LYON Taming. Journal of Finance, 1986, 41(3): 561-576.

[5] Merton, R.C. On the Pricing of Corporate Debt: The Risk Structure of Interest Rates. Journal of Finance, 1974, 29(2): 449-470.

[6] Nyborg, K. G., 1996, The use and pricing of convertible bonds, Applied Mathematical Finance 3, P167-190.

[7] Takahashi, A., T. Kobayashi, and N. Nakagawa, 2001, Pricing convertible bonds with default risk: A Duffie-Singleton Approach, Journal of Fix Income, December 2001, P20-29.

[8] Tsiveriotis, K., Fernandes, C.. Valuing Convertible Bonds with Credit Risk. Journal of Fixed Income, 1998, 8(3): 95-102.